INVARIANTS OF KNOTTED SURFACES FROM LINK HOMOLOGY AND BRIDGE TRISECTIONS

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Abstract. Meier and Zupan showed that every surface in the four-sphere admits a *bridge trisection* and can therefore be represented by three simple tangles. This raises the possibility of applying methods from link homology to knotted surfaces. We construct a invariant of a bridge-trisected surface in the form of an A_∞-algebra. Both invariants are defined by a novel connection between A_∞-algebras and Manolescu and Ozsváth's hyperboxes of chain complexes.

Introduction

A link homology theory is a gadget which assigns chain complexes to link diagrams so that the homology group of the complex is a link invariant. The ur-theory is Khovanov homology, whose Euler characteristic is the famous Jones polynomial. Khovanov's construction has spawned many variants and generalizations. All of these theories are expected to have a certain functoriality property: cobordisms of links should induce maps of homology groups which depend only on the isotopy class of the cobordism. This suggests that these theories should say something about closed surfaces in S^4, but the usual recipes have been shown to be ineffective.

In this paper, we apply link homology to Meier and Zupan's *bridge trisections* [16], a new perspective on knotted surfaces. Just as every link in S^3 can be split into two trivial tangles, so every surface-link in S^4 can be split into three trivial systems of disks. These systems meet along three tangles. The upshot is that every isotopy class of surface in S^4 can be represented by a trio of trivial tangles which pair together into unlinks. A diagram of three such tangles is called a *triplane diagram*.

What can link homology say about knotted surfaces through bridge trisections? More specifically,

- Can link-homological techniques produce invariants of knotted surfaces via bridge trisections?
- Can link-homological techniques produce obstructions to *destabilization* of triplane diagrams? (As for bridge splittings of links, destabilization is an

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operation which preserves the isotopy type of the surface but not the bridge trisection.)

There is some tension between these questions: the first asks for tools which are invariant under stabilization while the second asks for tools which detect stabilization.

In an earlier version of this paper, we answered the first question with a "yes." Due to a technical issue, this claim is no longer valid. We have removed that section from the present preprint.

Let $\mathbf{t} = (t_1, t_2, t_3)$ be a triplane diagram. Consider the group

$$A(\mathbf{t}) = \bigoplus_{i,j=1}^{3} \mathrm{CKh}(t_i \bar{t}_j)$$

where $\bar{t}_j$ denotes the mirror of t_j. The definition of triplane diagram ensures that each link $t_i \bar{t}_j$ is an unlink, so the homology of $A(\mathbf{t})$ is not interesting. This group can be given the structure of an associative algebra following Khovanov in [10]. In Section 3 we construct an A_∞-algebra by extending the construction to *Szabó homology* [24], a combinatorial link homology theory which interpolates between the world of link homology and Floer homology. Write $\mathrm{CSz}(\mathcal{D})$ for the Szabó chain group of the link diagram $\mathcal{D}$.

Theorem 2. *Let* $\mathbf{t}$ *be a triplane diagram for* $\mathcal{K}$*. Then*

$$\mathcal{A}(\mathbf{t}) = \bigoplus_{i,j=1}^{3} \mathrm{CSz}(t_i \bar{t}_j)$$

has the structure of an A_∞*-algebra over the ring* $\mathbb{F}[W]$*. Suppose that* $\mathbf{t}'$ *is a triplane diagram for* $\mathcal{K}$ *which presents the same bridge trisection as* $\mathbf{t}$*. Then* $\mathcal{A}(\mathbf{t})$ *and* $\mathcal{A}(\mathbf{t}')$ *are* A_∞*-chain homotopic.*

See Definition 3.5 and Theorem 4.1 for precise statements. $\mathcal{A}(\mathbf{t})$ is not an invariant of $\mathcal{K}$ up to isotopy – the chain homotopy type of $\mathcal{A}(\mathbf{t})$ changes under stabilization. Therefore it may be useful in obstructing destabilization.

Szabó homology and *Bar-Natan homology* were combined in [21] to produce a new link homology theory. The combined theory, which we call $\mathrm{CS}(\mathcal{D})$, is not as well-behaved as Szabó homology. The construction above produces some sort of "perturbed" A_∞-algebra. This structure still admits triple multiplication

$$\bar{\mu}_3 \colon \ \mathrm{CS}(t_i \bar{t}_{i+1}) \otimes \mathrm{CS}(t_{i+1} \bar{t}_{i+2}) \otimes \mathrm{CS}(t_{i+2} \bar{t}_i) \to \mathrm{CS}(t_i \bar{t}_i)$$

for $i \in \{1, 2, 3\}$.

Algebraic techniques. To prove Theorem 2 we use the language of *hyperboxes* developed by Manolescu and Ozsváth in [14] in the context of Heegaard Floer homology. To motivate the construction, let's first see how to build an associative algebra from a triplane diagram $\mathbf{t}$ ala Khovanov. Suppose that each tangle in $\mathbf{t}$ is the half-plat closure of a $2b$-strand braid. There is a multiplication map

$$\mathrm{CKh}(t_i\bar{t}_j) \otimes \mathrm{CKh}(t_j\bar{t}_k) \to \mathrm{CKh}(t_i\bar{\beta}_j\beta_j\bar{t}_k) \cong \mathrm{CKh}(t_i\bar{t}_k)$$

defined using the cobordism shown in Figure 1. The first step consists of b one-handle attachments and the second consists of Reidemeister 2 moves. Extend this to a map

$$\mu_2\colon\ A(\mathbf{t}) \otimes A(\mathbf{t}) \to A(\mathbf{t})$$

by linearity and the rule that μ_2 vanishes on summands of the form

$$\mathrm{CKh}(t_i\bar{t}_j) \otimes \mathrm{CKh}(t_k\bar{t}_\ell), \qquad j \neq k.$$

μ_2 is a chain map and therefore $A(\mathbf{t})$ is a differential graded algebra. Now suppose one wants to define a map

$$\mu_3\colon\ A(\mathbf{t})^{\otimes 3} \to A(\mathbf{t})$$

which is a chain homotopy between $\mu_2 \circ (\mathrm{Id} \otimes \mu_2)$ and $\mu_2 \circ (\mu_2 \otimes \mathrm{Id})$. (These two maps are equal in Khovanov homology – forget that for a moment.) μ_3 should represent an isotopy between the two cobordisms in Figure 2: it shuffles b one-handle attachments past b other one-handle attachments. So it is somehow built from b^2 smaller homotopies.

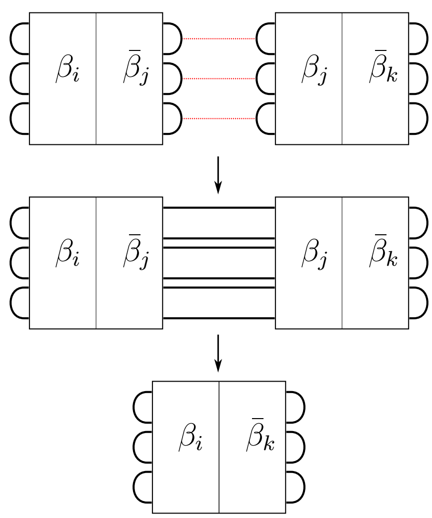

FIGURE 1. A cobordism from $t_i\bar{t}_j \coprod t_j\bar{t}_k$ to $t_i\bar{t}_k$.

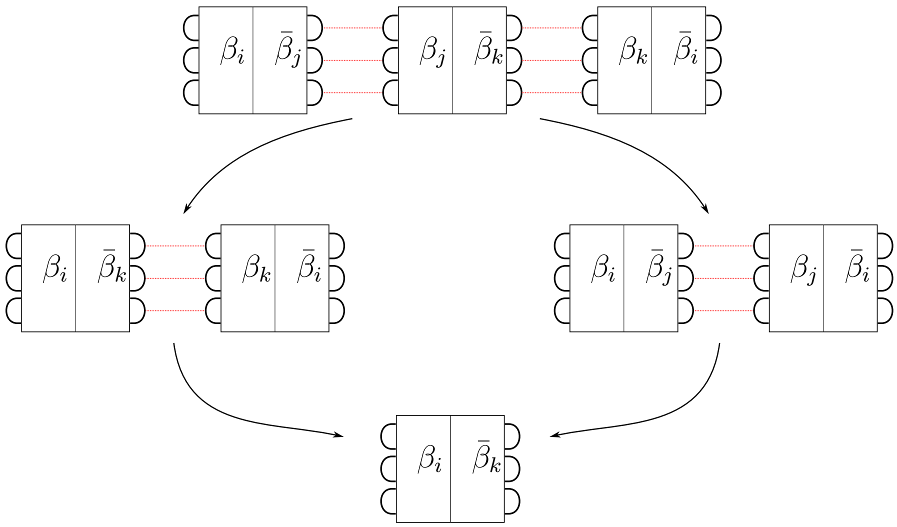

FIGURE 2. Two cobordisms from $t_i\bar{t}_j \coprod t_j\bar{t}_k \coprod t_k\bar{t}_i$ to $t_i\bar{t}_i$.

Hyperboxes are a way to organize these homotopies. Consider the diagram of chain complexes and chain maps

$$C_0 \xrightarrow[f_0]{} C_1 \xrightarrow[f_1]{} C_2 \xrightarrow[f_2]{} C_3.$$

This "factored mapping cone" contains at least as much information as the mapping cone of $f_2 \circ f_1 \circ f_0$. This a one-dimensional hyperbox of chain complexes. Now suppose that f_i is chain homotopic to some map f_i' for each i. A patient algebraic topology student can write down a chain homotopy between $f_2 \circ f_1 \circ f_0$ and $f_2' \circ f_1' \circ f_0'$. What if these homotopies come with their own factorizations? (For example when we move a one-handle past b other one-handles.) The result is a two-dimensional lattice of chain complexes, chain maps, and homotopies – a two-dimensional hyperbox of chain complexes.

One can turn a one-dimensional hyperbox into a mapping cone by simply composing the maps. Similarly, one can turn a two-dimensional hyperbox into a two-dimensional cubical complex. This process is called *compression*. The multiplication map $\mu_k\colon \mathcal{A}^{\otimes k}(\mathbf{t}) \to \mathcal{A}(\mathbf{t})$ is defined by setting up a k-dimensional hyperbox using $(k-1)$ families of one-handle attachments. The diagonal maps are defined using Szabó homology. The compression of this hyperbox is a k-dimensional cubical complex. The k-dimensional diagonal map is μ_k.

To show $\{\mu_i\}_{i=1}^{\infty}$ satisfies the A_∞-relations we must show that many hyperboxes are suitably related. We call such a collection of hyperboxes a *system of hyperboxes*. From a system of hyperboxes one can build an A_∞-algebra; for this observation I am indebted to John Baldwin and Cotton Seed. To prove Theorem 2, we make this construction functorial.

Theorem 3. *There is an interesting functor from the homotopy category of systems of hyperboxes to the homotopy category of A_∞-algebras.*

The precise statement is Theorem 9.19. The exact statement and proof have been banished to the final, self-contained section of the paper. Along the way we prove some other new results which may be interesting to the hyperbox *cognoscenti*. The ideas of this section are also key to [2].

Connections to Floer homology and trisections of four-manifolds. Bridge trisections are the knot-theoretic analogues of Gay and Kirby's *trisections of four-manifolds* [6]. The analogue of a triplane diagram is a *trisection diagram*. Let Σ be a compact surface of genus g. Let α, β, and γ be families of g curves so that each triple (Σ, α, β), (Σ, β, γ), (Σ, γ, α) is a Heegaard diagram for $\# S^1 \times S^2$. From this data one can build a trisected four-manifold, and every trisection can be represented in this way. Such diagrams (with a basepoint) appeared in the Ozsváth and Szabó's Heegaard Floer homology as *Heegaard triples* [17].

We take Theorems 1 and 2 as proof of concept for the usefulness of Heegaard Floer homology in application to trisections. Meier and Zupan show that a bridge trisection of $\mathcal{K}$ induces a trisection of $\Sigma(\mathcal{K})$, the double cover of S^4 branched along $\mathcal{K}$. Conjecturally, $\mathrm{Sz}(L) \cong \widehat{\mathrm{HF}}(\Sigma(-L)) \otimes \widehat{\mathrm{HF}}(S^1 \times S^2)$, where $\Sigma(-L)$ stands for the double cover of S^3 branched along the mirror of L ([24, 22]). This suggests that one should be able to construct an A_∞-algebra from a trisection of a four-manifold.

Ozsváth and Szabó use sequences of these diagrams to define an invariant $\Phi(X)$ of closed four-manifolds in [18]. Therefore it is reasonable to ask if $\Phi(X)$ can be computed directly from a trisection diagram for X. This circle of ideas has produced enthusiasm and skepticism among the Heegaard Floer faithful. We conjecture that $q(\mathcal{K})$ is related to the Ozsváth-Szabó four-manifold invariant of the double cover of S^4 branched along $\mathcal{K}$.

Computation. All of the link homology theories in this paper are computable by hand. The author hopes to have a computer program to assist in computations of $q(\mathbf{t})$ soon.

Signs. We work over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, see Remark 2.2 for more on signs.

Acknowledgments. I am grateful to Jeff Meier and Alex Zupan for introducing me to bridge trisections and for their many patient explanations of the subject's subtleties. I am thankful for Dave Gay's guidance and his even more patient explanations.

This project began while I was a student under John Baldwin. Without his advice and encouragement it would not have been completed. I am particularly grateful to him and Cotton Seed for suggesting the connection between A_∞-algebras and hyperboxes.

I am very grateful to Oleg Viro who pointed out a significant error in a previous version of this work. Fixing that error led to significant revision and improvement of the entire paper.

Thanks also to Peter Feller, Eli Grigsby, Matt Hedden, Adam Levine, Gordana Matić, William Olsen, and Mike Willis for helpful conversations.

1. Topological background

1.1. **Braids and plat closures.** Let β be a $2b$-strand braid. In this paper braid typically extend from left to right. For a tangle t write $\bar{t}$ for the mirror image of t, e.g. $\overline{\beta} = \beta^{-1}$. Let p_b be the crossingless tangle shown in Figure 3. The link $p_b\beta\overline{p}_b$ is called the *plat closure of* β. The $(0, 2b)$-tangle $p_n\beta$, also denoted $\widehat{\beta}$, is called the *half-plat closure of* β.

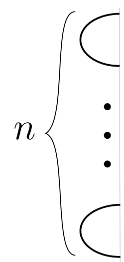

Figure 3. The tangle p_n.

The braid group B_{2b} acts (on the right) on the set of tangles with $2n$ ordered basepoints: $t \cdot \beta = t\beta$. The following definition is due to Hilden [7].

Definition. The *Hilden subgroup* $N_{2b} \subset B_{2b}$ is the stabilizer of p_b.

If $h, h' \in N_{2b}$ and $\beta \in B_{2b}$ then the plat closures of β and $h\beta h'$ are isotopic. Birman gave necessary and sufficient conditions for β and β' to have isotopic plat closures [4]. The situation for half-plat closures is much simpler.

Lemma 1.1. *Let $\beta, \beta' \in B_{2b}$. $\widehat{\beta}$ and $\widehat{\beta'}$ are isotopic if and only if $\beta'\beta^{-1} \in N_{2b}$.*

Proof. $p_b\beta' = p_b\beta$ if and only if $p_b\beta'\beta^{-1} = p_b$. □

1.2. **Knotted surfaces, bridge trisections, and triplane diagrams.** In the three-dimensional world, an *n-bridge sphere* for a knot K is a sphere S which cuts K into two trivial tangles so that $S \cap K$ is a collection of n points. "Trivial" means that all the arcs can be simultaneously isotoped to lie on S. Every trivial tangle may be written as the half-plat closure of a braid, and every half-plat closure of a braid is a trivial tangle. The four-dimensional equivalent of a trivial tangle is a trivial disk system.

Definition. A *trivial c-disk system* is a pair (X, C) where X is a four-ball and $C \subset X$ is a collection of c properly embedded disks which can be simultaneously isotoped to lie on ∂X.

A fundamental property of trivial disk systems is that (X, C) is determined up to isotopy rel boundary by the unlink $\partial X \cap C$, see [8]. So bisections of surfaces are not very interesting: the disk systems on each side must be identical. Gay and Kirby's trisections of four-manifolds [6] motivates the following definition of Meier and Zupan in [16].

Definition. A $(b; c)$*-bridge trisection* of a knotted surface $\mathcal{K} \subset S^4$ is a collection of three c-disk systems (X_1, C_1), (X_2, C_2), and (X_3, C_3), so that

- (X_1, X_2, X_3) is the standard genus 0 trisection of S^4. (See [6].)
- $C_1 \cup C_2 \cup C_3 = \mathcal{K}$.
- The tangle $T_{ij} = C_i \cap C_j$ is a trivial b-tangle in the three-ball $B_{ij} = X_i \cap X_j$ for all distinct i and j.

A $(b; c_1, c_2, c_3)$*-bridge trisection* is defined similarly, with (X_i, C_i) a trivial c_i-disk system.

Theorem ([16]). *Every knotted surface in S^4 admits a bridge trisection.*

See Figure **??** for a non-trivial example of a bridge trisection. The sphere at the core of the genus 0 trisection of S^4 is called the *bridge sphere*. The tangles T_{ij} intersect the bridge sphere in $2b$ points. The link $T_{ij}\overline{T}_{jk}$ is an unlink for any i,j, k. Any (cyclically ordered) triple of tangle diagrams (t_1, t_2, t_3) satisfying these two conditions is called a *triplane diagram*. By definition, every bridge trisection can be represented by a triplane diagram. From this perspective, it sometimes makes more sense to write c_{ij} for the number of components in $t_i\bar{t}_j$ rather than c_i. Meyer and Zupan show that every triplane diagram represents a bridge trisection and they determine a complete set of Reidemeister-type moves for triplane diagrams.

Theorem ([16]). *Two triplane diagrams represent isotopic surfaces if and only if they are related by a sequence of the following* triplane moves.

Interior Reidemeister move: *a Reidemeister move on any of the three tangles performed in the complement of a neighborhood of the bridge sphere.*

Braid transposition: *the addition of an Artin generator of the braid group B_{2b} or its inverse to the ends of all three tangles.*

Stabilization and destabilization: *Let i, j, and k be distinct. Suppose that $t_i\bar{t}_j$ has a crossingless component C. Let γ be an arc so that $\partial\gamma$ lies on C. Suppose that the interior of γ does not intersect $t_i\bar{t}_j$ and that γ meets the bridge sphere in a* *single point called p. The* stabilization of $\mathbf{t}$ along γ *is the result of surgering along γ to obtain two new tangles, t'_i and t'_j, then adding a small arc to t_k at p to obtain t'_k. Destabilization is the reverse process.*

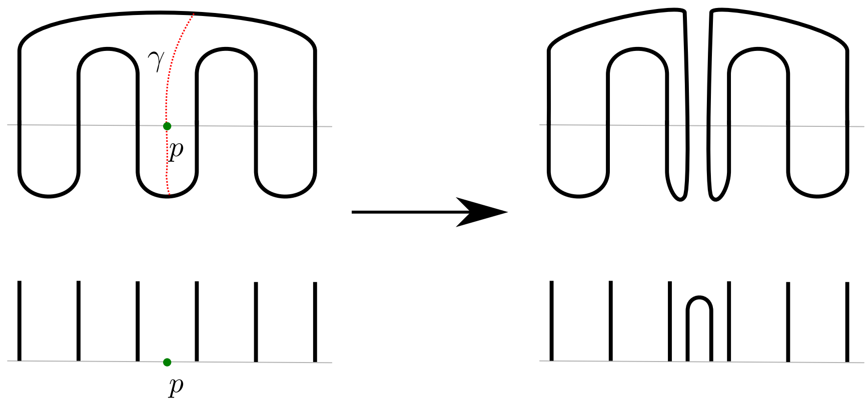

FIGURE 4. Stabilization along the arc γ. On top left, the crossingless component C of $t_i\overline{t_j}$. On the bottom left, the bottoms of the strands of t_k and the point p. On the right, the results of stabilization.

The first two moves correspond to isotopies of the trisection, i.e. isotopies of the surface through trisections which do not pass through the bridge sphere. Stabilization corresponds to pushing part of the surface through the spine and thus it changes the isotopy type of the trisection. In contrast to some other Reidemeister-type theorems, destabilization really is necessary: there are triplane diagrams for the same isotopy class of surface which are not isotopic after any number of stabilizations.

Some basic features of a knotted surface can be easily read off from a triplane diagram. For example, if $\mathbf{t}$ is a $(b; c_1, c_2, c_3)$-triplane diagram, then the Euler characteristic of the surface it represents is given by $c_1 + c_2 + c_3 - b$.

Definition. An *orientation* of the triplane diagram (t_1, t_2, t_3) is a choice of orientation on each tangle so that $t_1\overline{t_2}$, $t_2\overline{t_3}$, and $t_3\overline{t_1}$ are oriented as links.

An orientation of a triplane diagram can be packaged as an orientation of each of the $2b$ points which are common to the three tangles. The following proposition is straightforward.

Proposition. *Let $\mathbf{t}$ be a triplane diagram for $\mathcal{K}$. The set of orientations on $\mathcal{K}$ is in bijection with the set of orientations of $\mathbf{t}$.*

Suppose that $\tilde{\mathbf{t}}$ is a stabilization of the oriented triplane diagram $\mathbf{t}$. There is a unique orientation on $\tilde{\mathbf{t}}$ which extends the orientation on $\mathbf{t}$ in the sense that the $2b$

common points of $\tilde{\mathbf{t}}$ which are naturally identified with those of $\mathbf{t}$ have the same orientation as in $\mathbf{t}$.

2. Szabó's link homology theory

This section introduces Szabó homology following the presentation in [21]. (The theory first appeared in [24].) Write V for the algebra $\mathbb{F}[X]/(X^2)$. It is standard to write v_+ for 1 and v_- for X. For a crossingless link diagram $\mathcal{D}$ with k components, define

$$\mathrm{CKh}(\mathcal{D}) = V^{\otimes k}.$$

Concretely, $\mathrm{CKh}(\mathcal{D})$ is the vector space with a basis given by the labelings of the components of $\mathcal{D}$ by the symbols $+$ and $-$. These labelings are the *canonical generators* of $\mathrm{CKh}(\mathcal{D})$. Now suppose that $\mathcal{D}$ has c crossings. Figure 5 shows the two

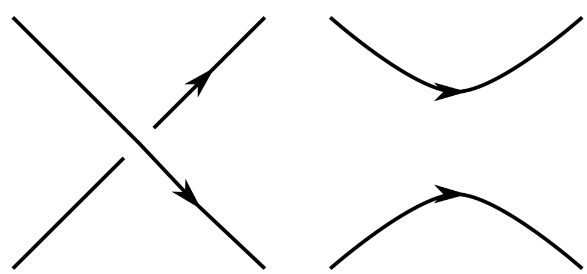

Figure 5. A crossing, its 0-resolution, and its 1-resolution.

ways to resolved a crossing. The set of resolutions of $\mathcal{D}$ is thus indexed by $\{0,1\}^c$ (after ordering the crossings). For $I \in \{0,1\}^c$ write $\mathcal{D}(I)$ for the resolution of $\mathcal{D}$ according to I. The collection of these diagrams is the *cube of resolutions*. The Khovanov chain group of $\mathcal{D}$ is defined as

$$\mathrm{CKh}(\mathcal{D}) = \bigoplus_{I \in \{0,1\}^c} \mathrm{CKh}(\mathcal{D}(I)).$$

One can define Khovanov homology over $\mathbb{F}[W]$ or $\mathbb{F}[U,W]$ where U and W are formal variables. The definition is exactly the same except that $\mathrm{CKh}(\mathcal{D}(I))$ is generated by $\mathbb{F}[U,W]$-linear combinations of the canonical generators.

There is a partial order on the cube of resolutions induced by the order on $\{0,1\}$. Write $\|I-J\|$ for the ℓ^∞ distance between I and J. If $I < J$, then $\mathcal{D}(J)$ may be obtained from $\mathcal{D}(I)$ by $\|I-J\|$ diagrammatic one-handle attachments. These one-handle attachments can be described by *surgery arcs* in $\mathcal{D}(I)$. An oriented surgery arc is called a *decoration*. A planar diagram with k decorations is called a *k-dimensional configuration*. Orienting the surgery arcs in the all-zeroes resolution I_0

of $\mathcal{D}$ orients them in every other resolution. We will always assume that orientations of decorations on other resolutions are induced in this way.

So for $I < J$ and $\|I - J\| = k$ there is a k-dimensional configuration $\mathcal{C}(I, J)$ which describes how to obtain $\mathcal{D}(J)$ from $\mathcal{D}(I)$. The collection of circles in $\mathcal{C}(I, J)$ which intersect decorations is the *active part* of $\mathcal{C}(I, J)$. The other circles form the *passive part*. We will often conflate these circles and their labels in canonical generators, e.g. in the next paragraph.

To define a link homology theory, one cooks up a map $F_{\mathcal{C}(I,J)} : \mathrm{CKh}(I) \to \mathrm{CKh}(J)$ for each I and J. Each of these maps acts by the identity on the passive part of $\mathrm{CKh}(I)$. This property is called the *extension rule*.

Definition 2.1. Let $\mathcal{C}(I, J)$ be a k-dimensional configuration from $\mathcal{D}(I)$ to $\mathcal{D}(J)$.

- The *Khovanov configuration map* $\mathfrak{K}_{\mathcal{C}}$ is defined via the Frobenius algebra structure on $\mathbb{F}[X]/(X^2)$. If $k > 1$ then $\mathfrak{K}_{\mathcal{C}} = 0$. If $k = 1$, then acts by multiplication or co-multiplication depending on the number of active circles.
- The *Szabó configuration map* $\mathfrak{S}_{\mathcal{C}}$ is defined in [24]. If $\mathcal{C}$ is one-dimensional then $\mathfrak{S}_{\mathcal{C}} = 0$. It is important that $\mathfrak{S}$ satisfies the *disconnected rule:* if the union of the active part of $\mathcal{C}$ and the decorations has more than one connected component, then $\mathfrak{S}_{\mathcal{C}} = 0$. If $\mathcal{C}$ has a degree one circle which is v_+-labeled in x, then $\mathfrak{S}_{\mathcal{C}}(x) = 0$.

$\mathrm{CKh}(\mathcal{D})$ has two gradings. Let $x \in \mathrm{CKh}(\mathcal{D}(I))$ be a canonical generator. The *homological* and *quantum gradings* of x are

$$
\begin{aligned}
h(x) &= \|I\| - n_- \\
q(x) &= \tilde{q}(x) + \|I\| + n_+ - 2n_-.
\end{aligned}
$$

These can be combined into the Δ-*grading* $h - q/2$. (This grading is typically called δ.) Let W be a formal variable with (h, q)-grading $(-1, -2)$. Note that the q-grading in Section 3 is slightly different.

For a link diagram $\mathcal{D}$ define

$$
\begin{aligned}
d_{\mathrm{Kh}}, d_{\mathrm{Sz}} \colon\ & \mathrm{CKh}(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D}) \\
d_{\mathrm{Kh}} &= \sum_{I<J} \mathfrak{K}_{\mathcal{C}(I,J)} \\
d_{\mathrm{Sz}} &= \sum_{I<J} W^{\|I-J\|-1} \mathfrak{S}_{\mathcal{C}(I,J)}
\end{aligned}
$$

Theorem ([24])**.** *$d_{\mathrm{Kh}} + d_{\mathrm{Sz}}$ is a differential of degree $(1, 0)$ on $\mathrm{CKh}(\mathcal{D})$. The graded chain homotopy type of $(\mathrm{CKh}(\mathcal{D}), d_{\mathrm{Kh}} + d_{\mathrm{Sz}})$ is a link invariant.*

2.1. **Cobordisms.** It is important to understand how CSz interacts with cobordisms of links. Let $\Sigma \subset \mathbb{R}^3 \times I$ be a properly embedded link cobordism from L_0 to L_1. Let

$\mathcal{D}_0$ and $\mathcal{D}_1$ be diagrams for L_0 and L_1.[1] Write Σ as a composition of *elementary cobordisms:* handle attachments, planar isotopies, and Reidemeister moves. To each of these cobordisms we assign a map. The map

$$F_\Sigma\colon \ \mathrm{CSz}(\mathcal{D}_0) \to \mathrm{CSz}(\mathcal{D}_1)$$

ought to be the composition of these elementary maps. Of course it is not clear *a priori* that F_Σ is independent of the decomposition of Σ into elementary pieces.

We now define maps for the elementary cobordisms. A 1-handle attachment can be specified diagrammatically by a planar arc γ with its endpoints on $\mathcal{D}_0$. Orient this arc. Put a crossing in $\mathcal{D}_0$ along γ as in Figure 6 and call the resulting diagram $\mathcal{D}$. The 0-resolution of the new crossing yields $\mathcal{D}_0$ and the 1-resolution yields $\mathcal{D}_1$. The complex $\mathrm{CS}(\mathcal{D})$ is, by definition, the mapping cone of a map $\mathfrak{h}_\gamma$ from $\mathrm{CS}(\mathcal{D}_0)$ to $\mathrm{CS}(\mathcal{D}_1)$. This is the map assigned to the 1-handle attachment along γ. Loosely, to compute $\mathfrak{h}_\gamma(x)$ for a canonical generator x, compute $(d_{\mathrm{Kh}} + d_{\mathrm{Sz}})$ on $\mathrm{CKh}(\mathcal{D})$ but insert γ as a decoration into each configuration.

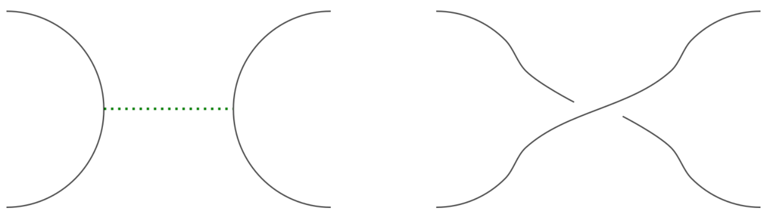

FIGURE 6. Adding a crossing along an arc.

0- and 2-handle attachments are much simpler. A 0-handle attachment adds a crossingless, closed component to a diagram. It is easy to show that

$$\mathrm{CSz}(\mathcal{D} \cup \circ) \cong \mathrm{CSz}(\mathcal{D}) \otimes \mathrm{CSz}(\circ).$$

The 0-handle attachment map is the map $\mathrm{CSz}(\mathcal{D}) \to \mathrm{CSz}(\mathcal{D} \cup \circ)$ induced by

$$x \mapsto x \otimes v_+$$

on simple tensors. The 2-handle attachment map is the dual map induced by

$$x \otimes v_- \mapsto x.$$

In [20] we showed that Szabó homology fits into the cobordism-theoretic framework developed by Bar-Natan to study Khovanov homology [3]. Bar-Natan assigns to each link diagram a complex in a certain cobordism category. Bar-Natan's Reidemeister invariance "maps" are sums of handle attachment maps. One upshot of [20] is that Bar-Natan's maps, interpreted as in the previous paragraphs, induce Reidemeister invariance maps on Szabó homology. For example, Bar-Natan's Reidemeister 2 map

[1] There is some subtlety here – see Section 2.3 of [1] – but it is not relevant to this paper.

is shown in Figure 6 of [3]. Take the Szabó chain group of each diagram to obtain $\mathrm{CSz}(\mathcal{D})$ (on top) and $\mathrm{CSz}(\mathcal{D}')$ (on the bottom). The sum of the cobordisms maps induces a map of Szabó complexes. This completes the definition of the elementary maps.

In [20] we showed that CSz is *functorial*: diagrammatic descriptions of isotopic cobordisms induce chain homotopic maps.[2] An important step in the proof is to show that one-handle attachment maps commute as long as the arcs of attachment are disjoint. These ideas will play an important part in this paper as well. Let γ and γ' be disjoint arcs of attachment. Write $\mathfrak{h}_{\gamma\cup\gamma'}$ for the map which counts configurations with both γ and γ'. Then

$$\mathfrak{h}_{\gamma} \circ \mathfrak{h}_{\gamma'} + \mathfrak{h}_{\gamma'} \circ \mathfrak{h}_{\gamma} = \mathfrak{h}_{\gamma\cup\gamma'} \circ \partial + \partial \circ \mathfrak{h}_{\gamma\cup\gamma'}.$$

This follows from studying the complex assigned to the diagram given by replacing both γ and γ' by crossings. The map $\mathfrak{h}_{\gamma}$ is defined via a mapping cone. We can view $\mathfrak{h}_{\gamma'}$ as a "map of mapping cones," one of whose components is $\mathfrak{h}_{\gamma\cup\gamma'}$. The entire complex is an *iterated mapping cone*. With n disjoint arcs of attachment one obtains an n-dimensional iterated mapping cone. This same argument with an analogous diagram can be used to show that Reidemeister 2 maps with disjoint support commute up to homotopy

Remark 2.2. Khovanov homology is only functorial in characteristic two. To extend the techniques of this and the next section to characteristic zero, one must first extend those of [20].

3. The A_∞-category of a bridge trisection

In this section we show how to construct an A_∞-cateogry $\mathcal{A}(\mathbf{t})$ from a triplane diagram $\mathbf{t}$. We build a *system of hyperboxes* from $\mathbf{t}$, then use Proposition 9.13 to obtain an A_∞-category. For details on systems of hyperboxes and their relationship to A_∞-categories see Section 9. Most of the hyperbox vocabulary in this Section comes from pages 37-39.

Definition 3.1. Let $\mathbf{t}$ be a triplane diagram. We say that $\mathbf{t}$ is in *plat form* if the three diagrams are half-plat closures of braids.

Every $(n, 0)$-tangle is isotopic to a half-plat closure. Therefore every trisection diagram may be put into plat form by interior Reidemeister moves.

Definition 3.2. Suppose that $\mathbf{t}$ is a triplane diagram in plat form. The *canonical surgery arcs* on

$$t_i\bar{t}_j \coprod t_j\bar{t}_k$$

[2] We actually proved this about CSz with $W = 1$, but the proof extends to the polynomial version without trouble.

are planar arcs connecting the plats in $\bar{t}_j$ and t_j, oriented towards t_j. Number these arcs with 1 to n from top to bottom. For a diagram of the form

$$t_{i_1}\bar{t}_{i_2} \coprod t_{i_2}\bar{t}_{i_3} \coprod \cdots \coprod t_{i_k}\bar{t}_{i_{k+1}}$$

There are $(k-1)$ *families of canonical surgery arcs*, defined and numbered similarly. They are the red, dotted arcs in Figure 7.

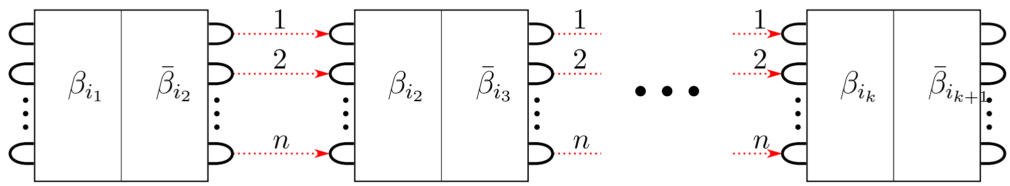

FIGURE 7

Let $s = (s_1, \ldots, s_{k+1})$ be a sequence of length $k+1 \geq 2$ in $\{1, 2, 3\}$. Define $\mathcal{D}_s$ to be the diagram

$$\mathcal{D}_s = (t_{s_1}\bar{t}_{s_2}) \coprod (t_{s_2}\bar{t}_{s_3}) \coprod \cdots \coprod (t_{s_k}\bar{t}_{s_{k+1}}).$$

For simplicity, let's first consider a triplane diagram in plat form with no crossings. (This implies that each tangle is the plat closure of the identity braid, but we won't use that fact.) Observe that there are $k-1$ families of canonical surgery arcs in $\mathcal{D}_s$. If $k = 1$, then set $H_s = \mathrm{CSz}(\mathcal{D}_s)$. If $k > 1$, then for a coordinate

$$\delta = (d_1, \ldots, d_{k-2}) \in [0, b]^{k-1}$$

let $\mathcal{D}_{s,\delta}$ be the diagram given by performing surgery along the first d_i arcs in the i-th family. For example, $\mathcal{D}_{s,(0,\ldots,0)} = \mathcal{D}_s$ and $\mathcal{D}_{s,(1,\ldots,1)} = t_{s_1}\bar{t}_{s_{k+1}}$. Let

$$C_s = \bigoplus_{\delta \in [0,b]^{k-2}} \mathrm{CSz}(\mathcal{D}_{s,\delta})\{-kb + \|\delta\|\}$$

where $\{-\}$ denotes a shift in the q-grading.

Fix a coordinate δ. A direction vector ϵ from δ picks out $\|\epsilon\|$ canonical arcs: for each i so that $\epsilon_i = 1$, take the (d_i+1)-st arc from the i-th collection. In other words, use the "next" arc from each family labeled by 1 in ϵ. Let $\mathcal{C}_{s,\delta,\epsilon}$ be the configuration whose underlying diagram is $\mathcal{D}_{s,\delta}$ and whose decorations are the arcs picked out by ϵ. Set

$$D^{\epsilon}_{\delta} \colon \ \mathrm{CSz}(\mathcal{D}_{s,\delta})\{-kb + \|\delta\|\} \to \mathrm{CSz}(\mathcal{D}_{s,\delta+\epsilon})\{-kb + \|\delta\| + \|\epsilon\|\}$$

$$D^{\epsilon}_{\delta} = \mathfrak{S}_{\mathcal{C}_{s,\delta,\epsilon}}$$

$$D_s = \sum_{\delta,\epsilon} D^{\epsilon}_{\delta}.$$

A coordinate vector δ and direction ϵ define a cubical complex which we call the (δ,ϵ)-*cube of* C_s. Therefore $H_s = (C_s, D_s)$ is a hyperbox of chain complexes with grading given by $\Delta = q - 2h$.

Remark 3.3. If we had shifted the *h-grading* by $\|\delta\|$, then the cube would be the Szabó complex of the link given by replacing each decoration from ϵ with a positive crossing. So instead of D^ϵ_δ having Δ-degree 1 like the Szabó differential it has degree is $1 - \|\epsilon\|$.

When $\mathbf{t}$ has crossings there are a few additional complications. First, the diagram $\mathcal{D}_{s_{\epsilon_1}}$ is isotopic to $t_1\overline{t_n}$ but not equal to it. One such isotopy is given by a sequence of Reidemeister 2 moves which cancel inverse Artin generators. For a fixed braid word β_i order these from the inside out: if $\beta = \sigma_{i_1}\cdots\sigma_{i_j}$, then

$$\beta^{-1}\beta = \sigma_{i_j}^{-1}\cdots\sigma_{i_1}^{-1}\sigma_{i_1}\cdots\sigma_{i_j}.$$

and the first cancellation is between $\sigma_{i_1}^{-1}$ and σ_{i_1}. Suppose that the braid word underlying t_i has length ℓ_i. Form a $(k-1)$-dimensional hyperbox of link diagrams of size

$$(b + \ell_{i_1}, \ldots, b + \ell_{i_{k-1}}).$$

For each $\delta = (d_1, \ldots, d_{k-1})$ there is a diagram $\mathcal{D}_{s,\delta}$ given as follows: if $d_i \leq n$, then perform surgery along the first d_i canonical arcs between $\overline{t_{i+1}}$ and t_{i+2}, just as in the crossingless case. If $d_i = b + m_i$ with $m_i > 0$ then perform all n surgeries, then perform the first m_i Reidemeister 2 moves. For such a δ define

$$\lfloor\delta\rfloor = \|(\min\{\epsilon_1, d_1\}, \ldots, \min\{\epsilon_{k-1}, d_{k-1}\})\|.$$

Set

$$C_s = \bigoplus_{\epsilon\in(d_1+c_1,\ldots,d_n+c_n)^{k-2}} \mathrm{CSz}(\mathcal{D}_{s,\epsilon})\{-kb + \lfloor\delta\rfloor\}.$$

Suppose that all of the coordinates of δ are less than b. The (δ,ϵ)-cube defines a diagram $\mathcal{D}_{\delta,\epsilon}$ by replacing the canonical surgery arcs with with positive crossings as in the crossingless case. These edge maps count configurations using the canonical surgery decoration along with any number of “internal” decorations from the crossings of $t_{s_i}\overline{t}_{s_{i+1}}$. The argument above shows that they have Δ-degree $\|\epsilon\| - 1$.

If δ has some coordinates greater than or equal to n, then ϵ may pick out Reidemeister 2 moves to surgery arcs. The (δ,ϵ)-cube is an iterated mapping cone as described in Section 2.1. The map D^ϵ_δ is a component of the iterated mapping cone of these moves.

Proposition 3.4. *Let* $\mathbf{t}$ *be a triplane diagram in plat form. The recipe above defines a system of hyperboxes of chain complexes* $\mathcal{H}(\mathbf{t})$ *over*

$$C = \bigoplus_{i,j=1}^{3} \mathrm{CSz}(t_i\bar{t}_j)\{-b\}.$$

graded by $\Delta = q - 2h$.

Proof. We have shown that each H_s is a hyperbox. We must check that the assignment $s \mapsto H_s$ defines a system of hyperboxes. Certainly the ϵ-corner of H_s is $C_{s(\epsilon)}$, ignoring the gradings. Let's check the gradings: suppose that C_s has dimension $(k-1)$. Then

$$C_{s,\delta} = \left(\bigotimes \mathrm{CSz}(t_{s_{i_j}}\overline{t_{s_{i_{j+1}}}})\right)\{-kb + \lfloor \delta \rfloor\}.$$

for some subsequence of s with length $k - \lfloor \delta \rfloor / b + 1$. Therefore there are $k - \lfloor \delta \rfloor / b$ factors in $C_{s,\delta}$, and

$$C_{s,\delta} = \bigotimes \mathrm{CSz}(t_{s_{i_j}}\overline{t_{s_{i_{j+1}}}})\{-b\}$$

as required.

The face condition follows from the extension rule and the disconnected rule. Let F be the face of H_s between the ϵ- and ϵ'-corners for some direction vectors ϵ and ϵ'. The extension rule implies that the map assigned to a handle attachment along a canonical surgery arc acts as the identity on the fixed sequence of s. Let c and c' be distinct elements of $c(\epsilon, \epsilon')$. Let γ and γ' be canonical surgery arcs which are attached as part of c and c', respectively. A configuration in F involving both γ and γ' must be disconnected. So there is a hyperbox F' so that

$$F \cong H_c \otimes F'.$$

Therefore H_s satisfies the face condition, cf. Lemma 9.10, part (2). □

Definition 3.5. Let $\mathcal{A}(\mathbf{t})$ be the A_∞-category over $\mathbb{F}[W]$ constructed from Propositions 3.4 and 9.13. The objects are the numbers 1, 2, and 3. We have $\mathrm{Hom}(i,j) = \mathrm{CSz}(t_i\overline{t_j})\{-b\}$. For

$$y \in \mathrm{Hom}(i_0, i_1) \otimes \cdots \otimes \mathrm{Hom}(i_{n-1}, i_n)$$

$\mu_k(y)$ is the image of y under the longest diagonal map of $\widehat{H_s}$.

Remark 3.6. The argument of this section is almost entirely formal; the only references to the specific structure of Szabó homology are the extension and disconnected rules. We therefore expect the argument to hold in any *conic, strong Khovanov-Floer theory* [20, 1], for example the Heegaard Floer homology of branched double covers of links.

3.1. **An example: the unknot.** Let us compute $\mathcal{A}(\mathbf{t})$ in the case that $\mathbf{t}$ is the crossingless, bridge number 1 triplane diagram for the unknotted sphere in S^4. $\mathrm{Hom}(i,j)$ has rank two and $\mu_1 = 0$.

Let x be a simple tensor of length $k > 2$. Let H_x be the hyperbox underlying $\mu_k(x)$. The active part of any connected configuration which appears in H_x consists of k circles connected to each other in a line. The map assigned to such a configuration is zero. The map assigned to a disconnected configuration is zero. We conclude that there are no non-zero configurations of dimension greater than one, and therefore all diagonal maps in H_x are zero. It follows that $\mu_k = 0$ for $k > 2$.

Let $\tilde{\mathbf{t}}$ be a stabilization of $\mathbf{t}$. One can show that μ_3 does not vanish on $\mathcal{A}(\tilde{\mathbf{t}})$. So stabilization can dramatically change the character of $\mathcal{A}$. Note also that the total rank of $\mathcal{A}(\tilde{\mathbf{t}})$ and its homology is greater than that of $\mathcal{A}(\mathbf{t})$ and its homology, respectively.

4. Invariance of the category

Theorem 4.1. *Let $\mathbf{t}$ be a triplane diagram in plat form. The A_∞-chain homotopy type of $\mathcal{A}(\mathbf{t})$ is an invariant of the trisection presented by $\mathbf{t}$.*

Proof. Let $\mathbf{t}'$ be a triplane diagram in plat form for the same bridge trisection. The remainder of this section is dedicated to showing that $\mathcal{A}(\mathbf{t}) \simeq \mathcal{A}(\mathbf{t}')$ if $\mathbf{t}$ and $\mathbf{t}'$ differ by a

- braid isotopy (Proposition 4.3)
- Hilden moves (Proposition 4.4)
- bridge sphere transposition (Proposition 4.5)

In light of Meier and Zupan's invariance statement (page 7), this suffices to prove the theorem. We construct a map of systems for each move and its reverse so that these maps are homotopy inverses. By Theorem 9.19, these two maps amount to a chain homotopy equivalence of A_∞-categories. □

The following lemma will also be useful, especially in concert with Proposition 9.23.

Lemma 4.2. *Let $\mathcal{D}$ be a link diagram. Suppose that $\mathcal{D}'$ is a subset of $\mathcal{D}$ which is a braid. Suppose further that $\mathcal{D}'$ is isotopic, as a braid, to the identity braid. Write $\mathcal{D}''$ for the link diagram which results from replacing $\mathcal{D}'$ with the identity braid.*

Let R and R' be two sequences of Reidemeister moves supported in $\mathcal{D}'$ which transform $\mathcal{D}'$ into the identity braid. R and R' induce maps

$$F_R \colon \ \mathrm{CSz}(\mathcal{D}) \to \mathrm{CSz}(\mathcal{D}'')$$
$$F_{R'} \colon \ \mathrm{CSz}(\mathcal{D}) \to \mathrm{CSz}(\mathcal{D}'')$$

These maps are chain homotopic. The homotopy can be described by a cobordism whose support lies in $\mathcal{D}'$.

Proof. This statement holds if one replaces $\mathrm{CSz}(\mathcal{D})$ by $[\![\mathcal{D}]\!]$, Bar-Natan's cobordism-theoretic link homology theory, see Lemmas 8.6 and 8.9 of [3]. Proposition 5.5 of [20] states that each conic, strong Khovanov-Floer theory "factors through" $[\![-]\!]$: there is a functor $\mathcal{F}_{\mathrm{CSz}}$ so that $\mathcal{F}_{\mathrm{CSz}}([\![\mathcal{D}]\!]) = \mathrm{CSz}(\mathcal{D})$. This functor, which was essentially defined in Section 2.1, carries handle attachments to handle attachment maps and Reidemeister moves to Reidemeister maps. So any relation that holds between these maps on $[\![\mathcal{D}]\!]$ holds for $\mathrm{CSz}(\mathcal{D})$. □

Proposition 4.3. *Let β and β' be braid words which represent equal elements of B_{2b}. Let $\mathbf{t} = (\widehat{\beta}, \widehat{\beta_2}, \widehat{\beta_3})$ and $\mathbf{t}' = (\widehat{\beta'}, \widehat{\beta_2}, \widehat{\beta_3})$. Then $\mathcal{A}(\mathbf{t}) \simeq \mathcal{A}(\mathbf{t}')$.*

Proof. It suffices to prove the theorem in the case that β and β' differ by a single relation in B_{2b}: commuting two Artin generators, canceling two Artin generators, or a triple-point move.

We begin with commuting generators. Let s be a sequence in $\{1, 2, 3\}$. Write H_s and H'_s for the hyperboxes assigned to s by $\mathcal{H}(\mathbf{t})$ and $\mathcal{H}(\mathbf{t}')$. If s does not include 1, then $H_s = H'_s$. If s does, then the maps on corresponding edges are identical except that the order of the cancellations on the commuted crossings has been swapped. Proposition 4.2 implies that $\mathcal{H}(\mathbf{t})$ and $\mathcal{H}(\mathbf{t}')$ are internally homotopic. Proposition 9.23 implies that $\mathcal{A}(\mathbf{t}) \simeq \mathcal{A}(\mathbf{t})$.

Suppose that β and β' differ by a single cancellation. Without loss of generality suppose that β' has two more crossings than β, and call the added crossings *new*. Let s be a sequence which contains 1. Then H'_s is larger than H_s: any axis which corresponds to 1 is longer than the same axis in H_s because it includes new Reidemeister 2 moves. The other edges of H'_s all correspond to edges of H_s as in the previous case. Elementarily enlarge H_s so that it has the same shape as H'_s and so that corresponding pieces line up. (In other words, the new cancellations in H'_s have the same coordinates as the elementary enlargements in H_s.) Abuse notation by calling this new hyperbox H_s.

Define a map $\rho_s\colon H_s \to H'_s$ cube-by-cube as follows. Let C and C' be corresponding cubes of H_s and H'_s. C belongs to some face F of C_s, and likewise for C'. Suppose for the moment that the contraction sequence of F has a single element and that C has no fixed sequence.[3] There are three possibilities.

- Neither C nor C' involves a canceling any crossings. In other words, $C = \mathrm{CSz}(\mathcal{D})$ and $C' = \mathrm{CSz}(\mathcal{D}')$ where $\mathcal{D}'$ may differ from $\mathcal{D}$ by some Reidemeister 2 moves. Define $\rho_s|_C$ to be the composition of these Reidemeister 2 maps.
- C and C' do involve undoing some crossings, but not the new ones. C and C' are (iterated) mapping cones of Reidemeister 2 maps on $\mathrm{CSz}(\mathcal{D})$ and $\mathrm{CSz}(\mathcal{D}')$, where $\mathcal{D}$ and $\mathcal{D}'$ are link diagrams which differ by Reidemeister 2 moves.

[3] In other words, the terminal vertex of F has group $\mathrm{CSz}(t_i\bar{t}_j)$ for some i and j.

Define $\rho_s|_C$ to be the cone of the new Reidemeister 2 map on the mapping cone. (In other words, iterate the mapping cone again).

- C' undoes some of the new crossings. Define $\rho_s|_C$ by the schematic in Figure 8. Each vertex represents an $(n-1)$-dimensional cube, the Szabó complex of a link. A portion of that link is shown. The horizontal axis is an axis of C_s and C'_s which corresponds to a 1 in s, so each horizontal arrow represents a map in C_s or C'_s. The Reidemeister moves on the solid arrows define maps of complexes. It follows from functoriality of Szabó homology that the squares, without the diagonal maps, commute up to homotopy. The homotopies, which are indicated on the schematic by the dotted, diagonal arrows, are also given by sums of handle attachment maps. $\rho_s|_C$ is the sum of the vertical and diagonal arrows. If C' undoes multiple pairs of new crossings, then define $\rho_s|_C$ by iterating Figure 8 like an iterated mapping cone.

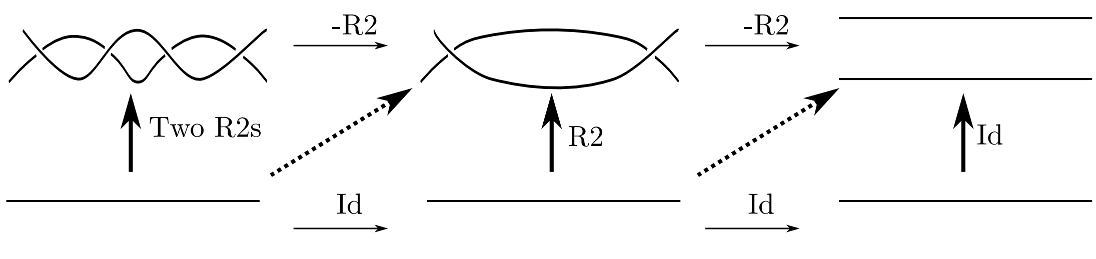

FIGURE 8

In general, C lives in a face whose contraction sequence c has multiple elements. Write f for the fixed sequence. By definition,

$$C = \left(\bigotimes_{c' \in c(\epsilon,\epsilon')} C_{c'} \right) \otimes \left(\bigotimes_{f' \in f} C_f \right).$$

Define $\rho_s|_C$ by

$$\rho_S = \left(\boxtimes_{c' \in c} \rho_{c'} \right) \boxtimes \left(\boxtimes_{f' \in f} \rho_{f'} \right).$$

Define ρ'_s by reversing all the Reidemeister 2 moves in the definition of ρ_s. We aim to show that $\rho_s \circ \rho'_s \simeq \mathrm{Id}$ by a chain homotopy J_s. If $\mathcal{D}_s$ is unchanged by the Reidemeister 2 moves, then ρ_s and ρ'_s are identity maps and $J_s = 0$. If s has a one-element contraction sequence and no fixed sequence then $\rho' \circ \rho$ is, on each cube C, the composition of a Reidemeister map and its inverse. Define J_s on C as the identity plus the homotopy j between $\rho_s|_C \circ \rho'_s|_C$ and Id_C, see Figure 9.

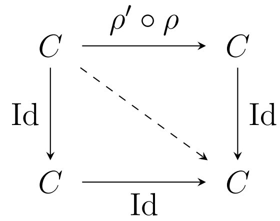

FIGURE 9. J_s is the sum of the vertical and diagonal maps.

If F has a larger contraction sequence then define $J_s|_F$ by equation (16) with $F_{s''_i} = (\rho' \circ \rho)_{s''_i}$ and $G_{s''_{i+1}} = \mathrm{Id}_{s''_{i+1}}$. In other words,

$$J_s|_F = \left(\boxtimes_{\substack{s' \in c \\ s' = (s''_1, \ldots, s_r)}} \bigoplus_{i=1}^{r} \Big((\rho' \circ \rho)_{s''_1} \boxtimes \cdots \boxtimes (\rho' \circ \rho)_{s''_{i-1}} \boxtimes J_{s''_i} \boxtimes \mathrm{Id}_{s''_{i+1}} \boxtimes \cdots \boxtimes \mathrm{Id}_{s''_r} \Big) \right) \otimes \mathrm{Id}\,.$$

Here $J_{s''_i}$ is the map for the singleton contraction sequence (s''_i). By Definition 9.18 J_s constitutes a chain homotopy between Id and $\rho' \circ \rho$.

We now turn to the triple point move. If s does not contain 1 then $\rho_s = \mathrm{Id}$. For other s, H_s and H'_s have the same size. Call the crossings affected by the triple point move *altered*. It is straightforward to define maps between cubes which do not cancel altered crossings.

The Reidemeister 3 move shuffles the order of the altered crossings. To correct for this, elementarily enlarge H_s on each axis which involves an altered crossing immediately before that crossing is to be canceled. Consider the hyperbox H''_s in which the elementary extension is changed to a pair of Reidemeister 3 moves on the affected crossings and the maps after the extension are changed to agree with those of H'_s. By Proposition 4.2, $H''_s \simeq H_s$ and therefore the systems $\mathcal{H}$ and $\mathcal{H}''$ are internally homotopic.

Elementarily enlarge H'_s in the same positions. Construct a map $\rho\colon H'_s \to H''_s$ using the Reidemeister 2 recipe: if corresponding cubes C and C' do not involve undoing altered crossings, then define $\rho_s|_C$ using the Reidemeister 3 maps. For cubes which undo altered crossings, the diagrams agree (near those crossings) and so the map can be defined using the same recipe. In the extended region, the map is defined by the two-dimensional schematic in Figure 10. □

Proposition 4.4. *Suppose that* $\widehat{\beta}$ *and* $\widehat{\beta'}$ *differ by a Hilden move. Let* $\mathbf{t} = (\widehat{\beta}, \widehat{\beta_2}, \widehat{\beta_3})$ *and* $\mathbf{t}' = (\widehat{\beta'}, \widehat{\beta_2}, \widehat{\beta_3})$*.* $\mathcal{A}(\mathbf{t})$ *is chain homotopic to* $\mathcal{A}(\mathbf{t}')$*.*

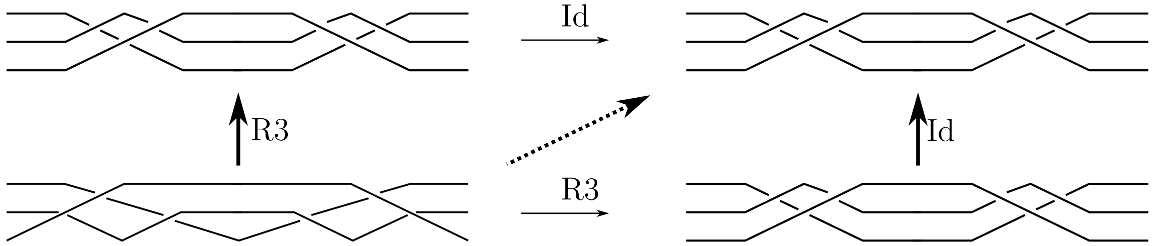

FIGURE 10. The schematic for the map ρ in the extension region. The top is H'' and the bottom is H'. In this case the square is commutative on the nose so no homotopy is necessary.

Proof. There is some h in the Hilden subgroup $N_{2n} \subset B_{2n}$, see Section 1.1, so that βh is isotopic to β'. There is a diagrammatic cobordism

$$\widehat{\beta} \to \widehat{\beta h}$$

whose support is disjoint from β. This follows from the motion group interpretation of N_b, see [5]. This cobordism is the composition of a sequence of Reidemeister moves, so it induces a chain homotopy equivalence of Szabó chain groups.

It suffices to consider the case in which h is one of Hilden's generators for N_b. The support of each generator is a small neighborhood of the plats, so the crossings added by the generator will always be canceled first. These canceling Reidemeister 2 moves have support disjoint from all the canonical surgery arcs (except for the ones which are pushed around by the Hilden move). Let $\mathcal{H}''$ be the system which is identical to $\mathcal{H}'$ except that the new cancellations happen immediately after the handle attachments between the affected plats. So the new order is: attach the plats above and including the ones affected by the Hilden moves, cancel the crossings introduced by Hilden moves, attach the rest of the plats, cancel the rest of the crossings. $\mathcal{H}''$ is internally chain homotopic to $\mathcal{H}'$.

Enlarge H_s so that it has the same shape as H''_s and so that the enlargements sit at the same positions as the cancellations of the new handles. We still call the result as H_s. We must cook up a map $H_s \to H'_s$ for each of Hilden's generators. The recipe is basically the same as in Proposition 4.3. The map is identity in the region "after" the new crossings have been canceled. In the region "before" the relevant plats are connected the map will be a composition of Reidemeister moves which realize the Hilden move. The interesting part of the map is the region in which the affected plats are connected and the new crossings are canceled.

The generator t_i is the easiest to work with. The downwards maps are Reidemeister maps. Consider the cobordism $\mathfrak{h} \circ r$, using the notation from Figure 11. This cobordism is isotopic to one which begins with a Reidemeister 2 move on (say) the left tangle, then connects the two plats. (This uses movie move 7 and 13 in [3].)

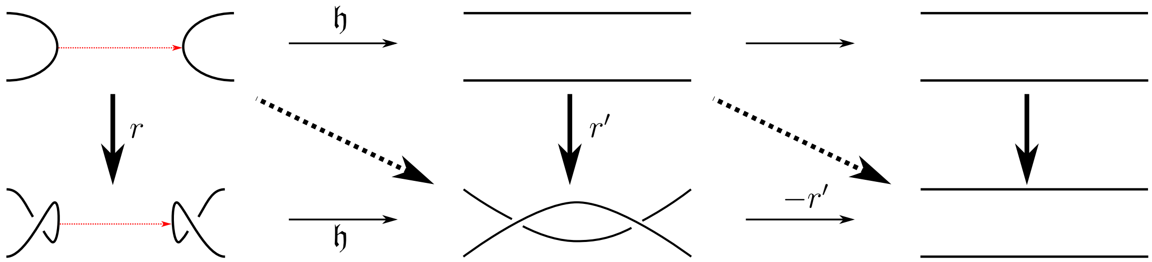

FIGURE 11. The interesting part of the map for t_i.

The support of the Reidemeister 2 move is disjoint from the surgery arc, so the maps assigned to those two cobordisms commute up to homotopy. This shows that $\mathfrak{h} \circ r \simeq r' \circ \mathfrak{h}$ in Bar-Natan's cobordism category. Therefore the homotopy can be chosen to be a map defined by handle attachments. This is the diagonal on the left of Figure 12. The diagonal on the right is the standard one which realizes $r \circ r^{-1} \simeq \mathrm{Id}$.

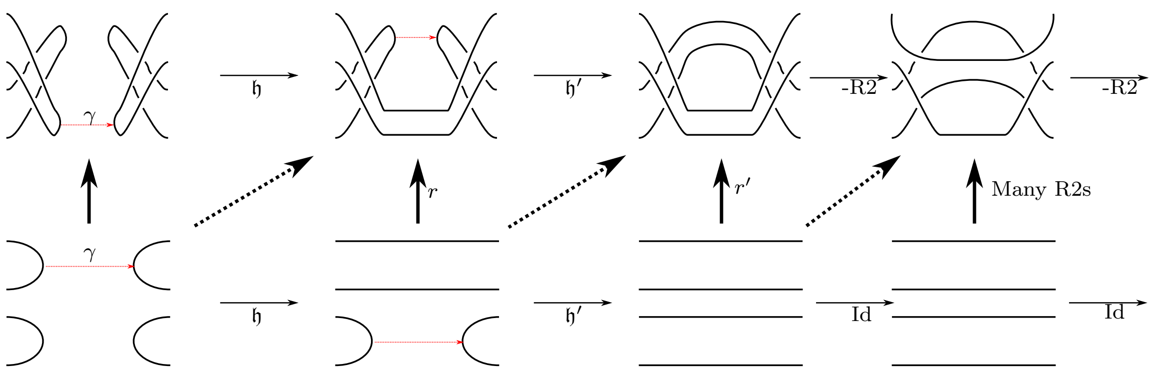

FIGURE 12

Figure 12 shows the argument for s_i. (We have changed the order of handle attachment, but of course the two systems are internally homotopic.) The first square commutes (up to homotopy) because the bottom plats can be passed under the upper plats by isotopies which are disjoint from γ. For the second square, repeatedly use movie move 15 to show that $\mathfrak{h}' \circ r$ is equivalent to the cobordism in Figure 13. This cobordism is a composition of Reidemeister 2 moves with a disjoint canonical handle attachment. Swap the order of these two maps to obtain $r' \circ \mathfrak{h}'$. The next square and all the ones after it commute up to homotopy by Proposition 4.2.

The arguments for the generators r_1 and r_2 are basically identical to s_i: use movie move 15 to write a cobordism as a sequence of Reidemeister 2 moves and disjoint canonical handle attachments.

Just as in the previous proof, these definitions amount to a map of hyperboxes $H_s \to H'_s$ for any sequence s. They constitute maps of systems of hyperboxes ρ

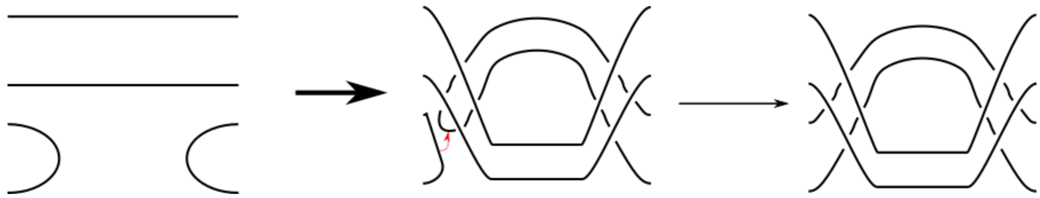

FIGURE 13

because they are defined by cobordisms. They are invertible, up to homotopy, because all of their components are Reidemeister moves: one can reverse the maps and run the same argument. Therefore $\rho' \circ \rho \simeq \mathrm{Id}$. The homotopies are also given by cobordisms, so together they form a homotopy between maps of systems. □

Proposition 4.5. *Suppose that* $\mathbf{t}$ *and* $\mathbf{t}'$ *are triplane diagrams in plat form which differ by a bridge sphere transposition. Then* $\mathcal{A}(\mathbf{t}) \simeq \mathcal{A}(\mathbf{t}')$.

The proof is essentially the same as Propositions 4.3 and 4.4 after observing that a bridge sphere transposition amounts to a Reidemeister 2 move on each link $t_i\bar{t}_j$.

5. Hybrid form and the homology algebra

5.1. **Hybrid form.** Every triplane diagram can be put into plat form, but this may increase the rank and complexity of $\mathcal{A}(\mathbf{t})$ significantly. For example, unknotted surfaces admit crossingless triplane diagrams, but the only crossingless diagrams in plat form are disjoint unions of unknotted spheres. In this section we show that $\mathcal{A}(\mathbf{t})$ can be constructed for a broad class of diagrams which include crossingless diagrams.

Definition 5.1. A triplane diagram is in *bridge form* if each tangle has no minima.

It follows that every strand in a bridge form diagram has a unique maximum, and so a bridge form diagram has canonical surgery arcs. Define $\mathcal{A}(\mathbf{t})$ as in the previous section with the stipulation that some crossings may need to be canceled before nested canonical one-handles are attached.

Proposition 5.2. *The construction of* $\mathcal{A}(\mathbf{t})$ *extends to diagrams in bridge form. Suppose that* $\mathbf{t}$ *and* $\mathbf{t}'$ *are diagrams in either form which present isotopic trisections. Then* $\mathcal{A}(\mathbf{t}) \simeq \mathcal{A}(\mathbf{t}')$.

Proof. $\mathcal{A}(\mathbf{t})$ is an A_∞-category by the same argument as Proposition 3.4. Any diagram in bridge form can be put into plat form by dragging the plats past some crossings. An argument like the proof of Proposition 4.4 (invariance under Hilden moves) shows that the resulting A_∞-category is A_∞-chain homotopy equivalent to the original. □

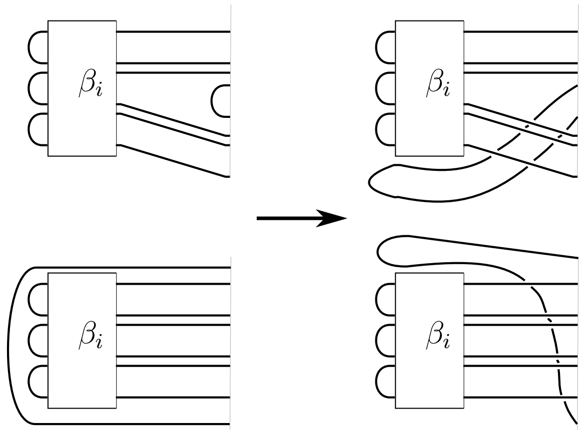

FIGURE 14. Transforming hybrid form into plat form.

5.2. **The homology cateory.** In this section we show that, for connected surfaces, the category $H(\mathcal{A}(\mathbf{t}))$ is essentially determined by the genus of $\mathcal{K}$.

Theorem 5.3. *Let* $\mathbf{t}$ *and* $\mathbf{t}'$ *be* $(b; c_{12}, c_{23}, c_{31})$ *triplane diagrams in bridge form which represent a connected surface. Then* $H(\mathcal{A}(\mathbf{t})) \cong H(\mathcal{A}(\mathbf{t}'))$ *as categories.*

Proof. Write A for $H(\mathcal{A}(\mathbf{t}))$ and write m for composition in A. The group $\mathrm{Kh}(t_i\bar{t}_j)$ has a unique generator of highest quantum grading called Θ_{ij}. Choose a basepoint on each component of $t_i\bar{t}_j$ for each i and j. There is a *basepoint action* on $\mathrm{CKh}(t_i\bar{t}_j)$, see [11], which satisfies

$$m((X_p)_*\Theta_{ij} \otimes \Theta_{jk}) = (X_p)_* m(\Theta_{ij} \otimes \Theta_{jk}).$$

In general the interaction between basepoint maps and Reidemeister maps can be complicated, but $t_i\bar{t}_j$ is always an unlink so

$$(\rho \circ X_p)_* = (X_{p'} \circ \rho)_*$$

where ρ is a Reidemeister map, p is a basepoint, and p' is a basepoint on the same component following the Reidemeister map. (This follows because $X_p \simeq X_{p'}$ when p and p' lie on the same component of a link). $\mathrm{Kh}(t_i\bar{t}_j)$ is a cyclic module over the ring

$$\Lambda_{ij} = \mathbb{F}[X_1, \ldots, X_{c_{ij}}]/(X_1^2, \ldots, X_{c_{ij}}^2)$$

with generator Θ_{ij}. If S is connected, then the cobordisms underlying m are connected for i, j, k distinct. It follows that

$$m((X_p)_*\Theta_{ij} \otimes \Theta_{jk}) = (X_p)_* m(\Theta_{ij} \otimes \Theta_{jk}) = (X_q)_* m(\Theta_{ij} \otimes \Theta_{jk}), \tag{1}$$

where q is any other basepoint. Therefore $m((X_p)_*\Theta_{ij} \otimes \Theta_{jk})$ is divisible by $X_1 \cdots X_{c_{ij}}$. From this and the degree of m it follows that

$$m(\Theta_{ij} \otimes \Theta_{jk}) = \sum_{j=1}^{c_{ik}} \left(\prod_{i=1}^{c_{ik}} X_1 \cdots \widehat{X}_j \cdots X_{c_{ik}} \right) \tag{2}$$

if it is non-zero (where the hat denotes omission).

Assume for a moment that $\mathbf{t}$ is in plat form. Consider the modules $\mathrm{CKh}(\beta_i \overline{\beta}_j)$ as defined by Khovanov in [10]. The map m is (a restriction of) the isomorphism

$$\mathrm{Kh}(\beta_i \overline{\beta}_j) \otimes_{H^{2b}} \mathrm{Kh}(\beta_j \overline{\beta}_k) \cong \mathrm{Kh}(\beta_i \overline{\beta}_k).$$

so $m(\Theta_{ij} \otimes \Theta_{jk}) \neq 0$. This argument adapts to bridge form diagrams without difficulty.

Now suppose that $\mathbf{t}$ and $\mathbf{t}'$ are both $(b; c_{12}, c_{23}, c_{31})$-trisections. Choose a bijection between the components of $t_i \bar{t}_j$ and $t'_i \bar{t'}_j$ for all i and j so that the bijection between $t_j \bar{t}_i$ and $t'_j \bar{t'}_i$ is given by mirroring. This induces a graded isomorphism $\mathrm{Kh}(t_i \bar{t}_j) \cong \mathrm{Kh}(t'_i \bar{t'}_j))$. Equations (1) and (2) show that this isomorphism extends to a functor with an inverse. □

Remark 5.4. Instead of studying $H(\mathcal{A}(\mathbf{t}))$, one could form a differential graded category $A(\mathbf{t})$ using the same construction as $\mathcal{A}(\mathbf{t})$ but with only the Khovanov differential. Equivalently, one could consider

$$\mathcal{A}(\mathbf{t}) \otimes_{\mathbb{F}[W]} \mathbb{F}[W]/(W).$$

Theorem 5.3 does not imply that $A(\mathbf{t})$ is determined by combinatorial data. It might be interesting to study Massey products on this algebra, particularly for surface-links.

Remark 5.5. The isomorphism above ignores some extra structure on $H(\mathcal{A}(\mathbf{t}))$. Instead of choosing a basepoint on each component, one could set the basepoints to be the bottoms of the tangles in $\mathbf{t}$. The ring

$$\Lambda = \mathbb{F}[X_1, \ldots, X_{2b}]/(X_1^2, \ldots, X_{2b}^2)$$

acts on the modules $\mathrm{Kh}(t_i \bar{t}_j)$. This action knows b and c_{ij}, c_{jk}, and c_{ki} and therefore it knows the genus of the underlying surface.

Remark 5.6. This theorem has precedents in independent work of Jacob Rasmussen [19] and Kokoro Tanaka [25]. There is a natural way to obtain an invariant of closed surfaces in S^4 from Khovanov homology: puncture the surface at its top and bottom, find a movie presentation Σ of the punctured surface, and compute the map

$$F_\Sigma \colon \ \mathrm{Kh}(\mathcal{U}) \to \mathrm{Kh}(\mathcal{U})$$

where U is the unknot. The Khovanov homology of U is generated by Θ and second homogeneous element Θ^-. The invariant of Σ is the Θ^--coordinate of $F_\Sigma(\Theta)$. Functoriality implies that this map does not depend on the particular movie presentation.

For grading reasons this map must vanish if $\mathcal{K}$ is not a torus, and Rasmussen and Tanaka showed that the map takes the same value on any torus. Tanaka proved a similar statement for Bar-Natan's deformation.

Remark 5.7. In Remark 9.21 we note that all of these A_∞-categories can be collapsed to A_∞-algebras. The results of this and the last section translate as follows: the A_∞-algebra, up to A_∞-chain homotopy, is an invariant of $\mathbf{t}$, and the homology algebra is not interesting for connected surfaces.

Theorem 5.3 and the following theorem of Kadeishvili allow us to make $\mathcal{A}(\mathbf{t})$ more concrete as an invariant.

Theorem (Kadeishvili)**.** *Let $\mathcal{A}$ be an A_∞-algebra. There is an A_∞-structure on $H(\mathcal{A})$ so that $\mu_1 = 0$, $\mu_2 = m_2^*$, and $\mathcal{A}$ is A_∞-quasi-isomorphic to $H(\mathcal{A})$.*

This theorem adapts immediately to A_∞-categories, see Remark 1.13 of [23]. An A_∞-category with $\mu_1 = 0$ is called *minimal*. From two triplane diagrams with the same combinatorial data, we obtain two minimal A_∞-category with isomorphic homology categories. So, fixing some combinatorial data, we may say that the invariant is an A_∞-structure (up to A_∞-chain homotopy) on a particular category. We explore this perspective in Section **??**

6. The Bar-Natan perturbation and the perturbed category

The Bar-Natan perturbation of Khovanov homology comes from replacing the Frobenius algebra $\mathbb{F}[X]/(X^2)$ by $\mathbb{F}[X,U]/(X^2 - U)$ for a formal variable U, see [3]. By setting $U = 0$ one recovers the Khovanov chain group. The variable U plays a similar role as W does in the Szabó differential: it keeps track of when one has "used" the Bar-Natan perturbation rather than the usual Khovanov differential.

Bar-Natan's differential does not commute with the Szabó differential, so one cannot simply add them together to get a omnibus link homology theory. Sarkar, Seed, and Szabó reconcile the theories in [21] by adding higher differentials ala Szabó homology. Recall from Section 2 that Szabó homology is defined via a configuration map $\mathfrak{S}$ which assigns linear maps to configurations. The Bar-Natan configuration map $\mathfrak{B}$, is defined as follows. Call a k-dimensional configuration a *tree* if it has k active starting circles and one active ending circle. Call it a *dual tree* if it has one active starting circle and k active ending circles. Let $x \in \mathrm{CKh}(\mathcal{D}(I))$ be a canonical generator. $\mathfrak{B}_{\mathcal{C}}(x) = 0$ unless $\mathcal{C}$ is a disjoint union of v_--labeled trees and v_+-labeled dual trees. On the factors belonging to a v_--labeled tree $\mathfrak{B}_{\mathcal{C}}$ is defined by

$$\mathfrak{B}_{\mathcal{C}}(v_- \otimes \cdots \otimes v_-) = v_-$$

and $\mathfrak{B}_{\mathcal{C}}(x) = 0$ for any other labeling. On the factors belonging to a v_+-labeled dual tree, define

$$\mathfrak{B}_{\mathcal{C}}(v_+) = v_+ \otimes \cdots \otimes v_+$$

and $\mathfrak{B}_{\mathcal{C}}(x) = 0$ for any other labeling. $\mathfrak{B}$ acts as the identity on factors belonging to passive circles. Define

$$d_{\mathrm{BN}} = \sum_{I<J} W^{\|I-J\|-1}\mathfrak{B}_{C(I,J)}.$$

If $C(I,J)$ is one-dimensional then $\mathfrak{B}_{C(I,J)}$ agrees with the usual Bar-Natan perturbation. Observe that d_{BN} preserves the homological grading but is increasing in the quantum grading.

Write $\partial' = d_{\mathrm{Kh}} + d_{\mathrm{Sz}} + d_{\mathrm{BN}}$ for the total differential and $\mathrm{CS}(\mathcal{D})$ for the resulting chain complex. Write $\mathrm{HS}(\mathcal{D})$ for its homology.

Theorem ([21])**:**

- $\mathrm{CS}(\mathcal{D})$ *is actually a chain complex.*
- *The quantum-filtered, homologically graded chain homotopy type of* $\mathrm{CS}(\mathcal{D})$ *and is a link invariant.*
- $\mathrm{HS}(\mathcal{D})$ *has a basis in canonical correspondence with the set of orientations of* $\mathcal{D}$*.*

The basis is described in Section **??**.

Remark 6.1. Sarkar, Seed, and Szabó define a *graded* link homology theory over $\mathbb{F}[U,W]$. We expect that many of the results in this section and the next can be reformulated and proven in the graded theory. However, the third bullet point above does not hold over $\mathbb{F}[U,W]$ for all links.

CS is not a strong Khovanov-Floer theory ([20]) because it does not satisfy the Künneth formula

$$\mathrm{CS}(\mathcal{D} \coprod \mathcal{D}') \not\simeq \mathrm{CS}(\mathcal{D}) \otimes \mathrm{CS}(\mathcal{D})$$

even though the two sides are isomorphic as $\mathbb{F}[W]$-modules. Also, if Σ and Σ' are cobordisms from $\mathcal{D}_0$ to $\mathcal{D}_1$ and $\mathcal{D}_0'$ to $\mathcal{D}_1'$, respectively, then

$$F_{\Sigma \coprod \Sigma'} \not\simeq F_\Sigma \otimes F_{\Sigma'}.$$

Nevertheless, we will show that the argument in [20] applies, *mutatis mutandis.*

Theorem 6.2. CS *is a functorial link invariant. The Reidemeister invariance maps are described by Bar-Natan's cobordism maps.*

Proof. CS satisfies every condition to be conic, strong Khovanov-Floer theory besides the Künneth formulae. If $\mathcal{D}'$ is crossingless – i.e. if $\mathrm{CS}(\mathcal{D}')$ has a vanishing differential – then

$$\mathrm{CS}(\mathcal{D} \coprod \mathcal{D}') \cong \mathrm{CS}(\mathcal{D}) \otimes \mathrm{CS}(\mathcal{D}')$$

as complexes. In the proof of functoriality for conic, strong Khovanov-Floer theories the Künneth formula for diagrams is only ever used in this situation. The Künneth formula for cobordisms is only used to prove the S, T, and $4Tu$ relations of [3]. In fact, it suffices that these relations hold for cobordisms of the form $\mathrm{Id} \otimes \Sigma'$ where Σ' is a cobordism of crossingless diagrams. It is straightforward to show that they hold in CS for totally crossingless diagrams.

Now

$$F_{\mathrm{Id} \coprod \Sigma'} = G_{\Sigma'} \otimes F_{\Sigma'}$$

where

$$G_{\Sigma'} \colon \ \mathrm{CS}(\mathcal{D}) \to \mathrm{CS}(\mathcal{D})$$

depends in some way on the topology of Σ'. If $F_{\Sigma'} = 0$, then $F_{\mathrm{Id} \coprod \Sigma'} = G_{\Sigma'} \otimes 0 = 0$. So the S, T, and $4Tu$ relations hold for CS, and this is all we need. □

6.1. **The perturbed A_3-category.** Let's try to apply the arguments of Section 3 to CS. There is a map

$$\mu_2' \colon \ \mathrm{CS}(t_i \bar{t}_j) \otimes \mathrm{CS}(t_j \bar{t}_k) \to \mathrm{CS}(t_i \bar{t}_k)$$

given by a sequence of one-handle attachment maps. Let $s = (s_1, s_2, s_3, s_4)$. Construct a hyperbox of chain complex H_s' exactly as in Section 3: each square is the CS complex of a link diagram. But this hyperbox cannot be part of a system of hyperboxes. For consider the bottom edge of H_s'. Label the relevant canonical surgery arcs $\gamma_1, \ldots, \gamma_b$. In Szabó homology, the composition of maps along this edge is

$$(\mathfrak{h}_{\gamma_b} \circ \cdots \circ \mathfrak{h}_{\gamma_1}) = (\mathfrak{h}_{\gamma_n} \circ \cdots \circ \mathfrak{h}_{\gamma_1})|_{\mathrm{CSz}(\mathcal{D}_{(s_1,s_2,s_3)})} \otimes \mathrm{Id}$$

because, for example,

$$\mathfrak{h}_{\gamma_1} = \left(\mathfrak{h}_{\gamma_1}|_{\mathrm{CSz}(\mathcal{D}_{(s_1,s_2,s_3)})} \right) \otimes \mathrm{Id}\,.$$

But $\mathfrak{B}$ assigns non-zero maps to *disjoint unions* of Bar-Natan configurations – it does not satisfy the disconnected rule. So in CS,

$$\mathfrak{h}_{\gamma_1} = \mathfrak{h}_{\gamma_1}|_{\mathrm{CS}(\mathcal{D}_{(s_1,s_2,s_3)})} \otimes \mathrm{Id} + \mathfrak{h}_{\gamma_1,\mathrm{BN}}|_{\mathrm{CS}(\mathcal{D}_{(s_1,s_2,s-3)})} \otimes W d_{\mathrm{BN}}$$

where $\mathfrak{h}_{\gamma,\mathrm{BN}}$ denotes the component of $\mathfrak{h}_\gamma$ counts Bar-Natan configurations. Loosely, attaching a one-handle to a diagram "activates" the Bar-Natan differential in other parts. The extra W comes from the fact that a k-dimensional Bar-Natan configuration is counted with coefficient W^{k-1}. Counting $(k+\ell)$-dimensional configurations by combining k- and ℓ-dimensional configurations separately under-counts by a factor of W. We will call this idea the *gluing principle*.

Therefore, in CS,

$$\begin{aligned}(\mathfrak{h}_{\gamma_n}\circ\cdots\circ\mathfrak{h}_{\gamma_1}) =& (\mathfrak{h}_{\gamma_1}|_{\mathrm{CS}(\mathcal{D}_{(s_1,s_2,s_3)})}\otimes \mathrm{Id} + \mathfrak{h}_{\gamma_1,\mathrm{BN}}|_{\mathrm{CS}(\mathcal{D}_{(s_1,s_2,s_3)})}\otimes W d_{\mathrm{BN}})\\ &\circ\cdots\circ\\ &(\mathfrak{h}_{\gamma_n}|_{\mathrm{CS}(\mathcal{D}_{(s_1,s_2,s_3)})}\otimes \mathrm{Id} + \mathfrak{h}_{\gamma_n,\mathrm{BN}}|_{\mathrm{CS}(\mathcal{D}_{(s_1,s_2,s_3)})}\otimes W d_{\mathrm{BN}})\end{aligned}$$

$$= (\mathfrak{h}_{\gamma_n}\circ\cdots\circ\mathfrak{h}_{\gamma_1})\otimes\mathrm{Id} + \sum_{i=1}^{n}(\mathfrak{h}_{\gamma_n}\circ\cdots\circ\mathfrak{h}_{\gamma_i,\mathrm{BN}}\circ\cdots\circ\mathfrak{h}_{\gamma_1})\otimes W d_{\mathrm{BN}} \tag{3}$$

(4)

It follows that the map induced by the bottom edge of H'_s is

$$\mu'_2\otimes\mathrm{Id} + \tilde{\mu}_2\otimes W d_{\mathrm{BN}}$$

for some map $\tilde{\mu}_2$. To be concrete, $\tilde{\mu}_2$ is a multiple of the component of μ'_2 which counts, at some point, a Bar-Natan configuration.

Define μ'_3 by compressing the hyperbox H'_s.

Definition 6.3. Let $\mathbf{t}$ be a triplane diagram in bridge form. Let $\mathcal{A}_S(\mathbf{t})$ be the vector space

$$\mathcal{A}_S(\mathbf{t}) = \bigoplus_{i,j=1}^{3}\mathrm{CS}(t_i\bar{t}_j)$$

equipped with the operations ∂, μ'_2, and μ'_3.

Proposition 6.4. $\mathcal{A}_S(\mathbf{t})$ *satisfies the following equations:*

$$\begin{aligned}\partial^2 &= 0\\ \partial\circ\mu'_2 + \mu'_2(\partial\otimes\mathrm{Id}) + \mu'_2(\mathrm{Id}\otimes\partial) &= W\mu'_2(d_{\mathrm{BN}}\otimes d_{\mathrm{BN}})\\ \mu'_3\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\partial + \mathrm{Id}\otimes\partial\otimes\mathrm{Id} + \partial\otimes\mathrm{Id}\otimes\mathrm{Id}) + \partial\circ\mu'_3 + \mu'_2(\mu'_2\otimes\mathrm{Id} + \mathrm{Id}\otimes\mu'_2)&\\ = W\mu'_2(\tilde{\mu}_2\otimes d_{\mathrm{BN}} + d_{\mathrm{BN}}\otimes\tilde{\mu}_2)&\\ + W\big(\mu'_3(d_{\mathrm{BN}}\otimes d_{\mathrm{BN}}\otimes\mathrm{Id}) + \mu'_3(d_{\mathrm{BN}}\otimes\mathrm{Id}\otimes d_{\mathrm{BN}})&\\ + \mu'_3(\mathrm{Id}\otimes d_{\mathrm{BN}}\otimes d_{\mathrm{BN}})\big) + W^2\mu'_3(d_{\mathrm{BN}}\otimes d_{\mathrm{BN}}\otimes d_{\mathrm{BN}})&\end{aligned}$$

We call these equations the *perturbed A_n-equations* for $n = 1, 2, 3$; compare with equation (8).

Proof. The first equation follows from the definition of CS. The second equation follows from the fact that μ'_2 is a chain map. The term on the right comes from the part of ∂ which does not respect the Künneth formula.

The third equation comes from the fact that $\widehat{H_s}$ is a chain complex for length four s. Each term represents part of the square of the differential from the initial vertex of $\widehat{H_s}$ to the terminal vertex. The terms on the right all come from the failure of the Künneth formula for CS. For the final term, Bar-Natan configurations are split into three pieces, so one must apply the gluing principle twice. □

Proposition 6.5. *Let* $\mathbf{t}$ *and* $\mathbf{t}'$ *be triplane diagrams in hybrid or plat form which differ by bridge sphere transpositions and interior Reidemeister moves. Then there is a collection of maps*

$$\rho'_i\colon\ \mathcal{A}_S(\mathbf{t})^{\otimes i} \to \mathcal{A}_S(\mathbf{t}')$$

so that ρ'_1 *is the direct sum of the Reidemeister maps on* CS. *These maps satisfy the following relations:*

$$\rho'_1 \circ \partial + \partial \circ \rho'_1 = 0. \tag{5}$$

$$\begin{aligned}\rho'_2(\partial \otimes \mathrm{Id}) + \rho'_2(\mathrm{Id} \otimes \partial) + \partial \rho'_2 + \mu'_2(\rho'_1 \otimes \rho'_1) + \rho'_1 \mu'_2 \\ = W\left(\rho'_2(d_{\mathrm{BN}} \otimes d_{\mathrm{BN}}) + \mu'_2 R_{1,1}\right)\end{aligned} \tag{6}$$

$$\begin{aligned}&\rho'_1\mu'_3 + \mu'_3(\rho_1 \otimes \rho_1 \otimes \rho_1) \\ &\quad = \rho'_3(\partial \otimes \mathrm{Id} \otimes \mathrm{Id}) + \rho'_3(\mathrm{Id} \otimes \partial \otimes \mathrm{Id}) + \rho'_3(\mathrm{Id} \otimes \mathrm{Id} \otimes \partial) \\ &\qquad + \partial\rho'_3 + \rho'_2(\mu'_2 \otimes \mathrm{Id}) + \rho'_2(\mathrm{Id} \otimes \mu'_2) + \mu'_2(\rho'_1 \otimes \rho'_2) + \mu'_2(\rho'_2 \otimes \rho'_1) \\ &\qquad + W\left(\rho'_3(d_{\mathrm{BN}} \otimes d_{\mathrm{BN}} \otimes \mathrm{Id}) + \rho'_3(d_{\mathrm{BN}} \otimes \mathrm{Id} \otimes d_{\mathrm{BN}}) + \rho'_3(\mathrm{Id} \otimes d_{\mathrm{BN}} \otimes d_{\mathrm{BN}})\right) \\ &\qquad + W\left(\rho'_2(\tilde{\mu}_2 \otimes d_{\mathrm{BN}}) + \rho'_2(d_{\mathrm{BN}} \otimes \tilde{\mu}_2)\right)) + W\left(\mu'_2(R_{2,1} + R_{1,2})\right). \\ &\qquad + W^2\rho'_3(d_{\mathrm{BN}} \otimes d_{\mathrm{BN}} \otimes d_{\mathrm{BN}})\end{aligned} \tag{7}$$

where

$$R_{1,1}\colon\ \mathcal{A}_S(\mathbf{t})^{\otimes 2} \to \mathcal{A}_S(\mathbf{t})^{\otimes 2}$$
$$R_{2,1}, R_{1,2}\colon\ \mathcal{A}_S(\mathbf{t})^{\otimes 3} \to \mathcal{A}_S(\mathbf{t}')^{\otimes 2}$$

are linear maps derived from $\rho'_2 \otimes \rho'_1$ *and* $\rho'_1 \otimes \rho'_2$.

The maps $R_{i,j}$ describe parts of the Reidemeister maps which do not respect the Künneth principle – their exact form is not important.

Proof. To prove Theorem 4.1 we showed constructed maps of systems of hyperboxes $\mathcal{H}(\mathbf{t}) \to \mathcal{H}(\mathbf{t}')$. For each sequence s we built a map of hyperboxes ρ_s. The A_∞-relations were derived from the fact that $\widehat{\rho_s}$ is a chain complex.

Write H_s and H'_s for the hyperboxes assigned to s by $\mathcal{A}_S(\mathbf{t})$ and $\mathcal{A}_S(\mathbf{t}')$. Define $\rho'_s\colon\ H_s \to H'_s$ using the same handle attachment maps as ρ_s. Define $\rho'_{|s|-1}$ to be the long diagonal in $\widehat{\rho_s}$. The relations satisfied by these maps stem from the fact that $\widehat{\rho_s}$ is a chain complex.

If s has length two then ρ_s is defined as the Reidemeister map $\mathrm{CS}(t_{s_1}\bar{t}_{s_2}) \to \mathrm{CS}(t'_{s_1}\bar{t'}_{s_2})$. This proves equation (6). For equation (7), study the two-dimensional cubical complex $\widehat{\rho_s}$. The terms on the right come from the Bar-Natan differential's disrespect for the Künneth principle. Define $R_{1,1}$ to be the part of $\rho_1 \otimes \rho_1$ which does not respect the Künneth principle.

Equation 7 follows from the same sort of reasoning. □

One could define a *perturbed A_∞-category* (resp. *perturbed A_∞-fuctor*) as a category (resp. functor) which follows all the relations which $\mathcal{A}_S(\mathbf{t})$ (resp. ρ') follows. In the context $U = W = 1$ all of these relations suggest that a perturbed A_∞-category is something like the mapping cone of an A_∞-functor f_{BN} with the property that $f_{\mathrm{BN}} \circ f_{\mathrm{BN}} = 0$. We would like to pursue this perspective further in future work. For this purposes of this paper we only need $n = 1$, 2, and 3 with coefficients in $\mathbb{F}[U,W]/(U-1) \cong \mathbb{F}[W]$

Definition 6.6. A *perturbed A_3-category* $\mathcal{A}'$ over $\mathbb{F}[W]$ consists of

- a set of objects $\mathrm{Ob}(\mathcal{A}')$.
- for every pair of objects $o, o' \in \mathrm{Ob}(\mathcal{A}')$ a $\mathbb{F}[U,W]$-module $\mathrm{Hom}_{\mathcal{A}'}(o,o')$.
- two differentials μ_1' and μ_1'' on $\mathrm{Hom}_{\mathcal{A}'}(o,o')$ for every $o, o' \in \mathrm{Ob}(\mathcal{A}')$ so that $\mu_1'\mu_1'' = \mu_1''\mu_1'$. Write $\mu_1 = \mu_1' + \mu_1''$.
- two maps $\mu_2', \mu_2''\colon\ \mathrm{Hom}_{\mathcal{A}'}(o,o') \otimes \mathrm{Hom}_{\mathcal{A}'}(o',o'') \to \mathrm{Hom}_{\mathcal{A}'}(o,o'')$ which satisfy some relations. Write $\mu_2 = \mu_2' + \mu_2''$. Then
$$\mu_2(\mu_1 \otimes \mathrm{Id}) + \mu_2(\mathrm{Id} \otimes \mu_1) + \mu_1\mu_2 = W\mu_2(\mu_1'' \otimes \mu_1'').$$
and
$$\mu_2'(\mu_1' \otimes \mathrm{Id}) + \mu_2'(\mathrm{Id} \otimes \mu_1') + \mu_1'\mu_2' = 0.$$
- two maps $\mu_3', \mu_3''\colon\ \mathrm{Hom}_{\mathcal{A}'}(o,o') \otimes \mathrm{Hom}_{\mathcal{A}'}(o',o'') \otimes \mathrm{Hom}_{\mathcal{A}'}(o'',o''') \to \mathrm{Hom}_{\mathcal{A}}(o,o''')$ which satisfy some relations. Write $\mu_3 = \mu_3' + \mu_3''$. Then
$$\begin{aligned}\mu_3(\mathrm{Id} \otimes \mathrm{Id} \otimes \mu_1 + \mathrm{Id} \otimes \mu_1 \otimes \mathrm{Id} + \mu_1 \otimes \mathrm{Id} \otimes \mathrm{Id}) + \mu_1\mu_3 + \mu_2(\mu_2 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mu_2)\\ = W\mu_2(\mu_2'' \otimes \mu_1'' + \mu_1'' \otimes \mu_2'')\\ + W\mu_3(\mu_1'' \otimes \mu_1'' \otimes \mathrm{Id} + \mu_1'' \otimes \mathrm{Id} \otimes \mu_1'' + \mathrm{Id} \otimes \mu_1'' \otimes \mu_1'')\\ + W^2\mu_3(\mu_1'' \otimes \mu_1'' \otimes \mu_1'')\end{aligned}$$
and
$$\mu_3'(\mathrm{Id} \otimes \mathrm{Id} \otimes \mu_1' + \mathrm{Id} \otimes \mu_1' \otimes \mathrm{Id} + \mu_1' \otimes \mathrm{Id} \otimes \mathrm{Id}) + \mu_1'\mu_3' + \mu_2'(\mu_2' \otimes \mathrm{Id} + \mathrm{Id} \otimes \mu_2') = 0.$$

Observe that the operations (μ_1', μ_2', μ_3') satisfy the usual A_∞-relations (see equation (8)).

Definition 6.7. Let $\mathcal{A}'$ and $\mathcal{B}'$ be peturbed A_3-algebras. A *perturbed A_3-functor* $f\colon \mathcal{A}' \to \mathcal{B}'$ consists of

- a map of sets $f\colon \mathrm{Ob}(\mathcal{A}') \to \mathrm{Ob}(\mathcal{B}')$.
- a chain map $f_1\colon \mathrm{Hom}_{\mathcal{A}'}(o, o') \to \mathrm{Hom}_{\mathcal{B}'}(f(o), f(o'))$ for every $o, o' \in \mathrm{Ob}(\mathcal{A}')$.
- a map $f_2\colon \mathrm{Hom}_{\mathcal{A}'}(o, o') \otimes \mathrm{Hom}_{\mathcal{A}'}(o', o'') \to \mathrm{Hom}_{\mathcal{B}'}(f(o), f(o'))$ which satisfies

$$f_2(\mu_{1,\mathcal{A}'}\otimes \mathrm{Id}) + f_2(\mathrm{Id}\otimes \mu_{1,\mathcal{A}}) + \mu_{1,\mathcal{B}'} + \mu_{2,\mathcal{B}}(f_1\otimes f_1) + f_1\mu_{2,\mathcal{A}} = W(f_1(\mu''_{1,\mathcal{A}}\otimes\mu''_{1,\mathcal{A}} + \mu'_{2,\mathcal{B}}(f'_1\otimes f'_1)))$$

  for some linear map f'_1.
- a map f_3 analogous to ρ'_3.

We leave the reader to define a chain homotopy equivalance of perturbed A_3-functors and summarize this section by the following theorem.

Theorem 6.8. *The perturbed A_3-chain homotopy type of $\mathcal{A}_S(\mathbf{t})$ with coefficients in $\mathbb{F}[W]$ is an invariant of the trisection presented by $\mathbf{t}$.*

7. Stabilization

Our goal in this section is to study the behavior of $\mathcal{A}_S(\mathbf{t})$ under stabilization. Throughout we take $\mathbf{t}$ to be a triplane diagram which admits a stabilization, $\tilde{\mathbf{t}}$ for that stabilization, and γ for the arc of stabilization. We assume without loss of generality that γ lies in $t_2\bar{t}_3$. Write $\gamma^\perp$ for the arc dual to γ in $\tilde{\mathbf{t}}$.

Proposition 7.1. *Let $\tilde{\mathbf{t}}$ be a stabilization of $\mathbf{t}$. There are maps*

$$\phi\colon \mathcal{A}_S(\mathbf{t}) \to \mathcal{A}_S(\tilde{\mathbf{t}})$$
$$\phi'\colon \mathcal{A}_S(\tilde{\mathbf{t}}) \to \mathcal{A}_S(\mathbf{t}).$$

With $U = 1$, ϕ_1 is injective and ϕ' is surjective.

Perhaps not surprisngly these maps will make heavy use of γ and $\gamma^\perp$. But we must be careful: $\tilde{\mathbf{t}}$ has more canonical handle attachments than $\mathbf{t}$. The construction and proof occupies the rest of this section.

Assume that γ is horizontal and that it does not intersect any of the canonical surgery arcs. There are two canonical surgery which are the lowest such arcs in their families which lie above γ. In fact they have the same position in their families. Suppose that this is the nth position.

Let s be a sequence in $\{1, 2, 3\}$. We have two hyperboxes H_s and $\tilde{H}_s$. First, elementarily extend H_s along each of its axes at n, starting from the first. (Elementary extension is defined of hyperboxes is defined at the end of Section 9.) Call the resulting hyperbox $H_{s,n}$. We will define a map $\phi_s\colon H_{s,n} \to \widetilde{H}_s$ vertex by vertex. There is some number (perhaps zero) of copies of γ in the diagram at the initial vertex of $C_{s,n}$. Write γ_i for the copy of γ, if there is one, in $t_{s_i}\bar{t}_{s_{i+1}}$. Let $\delta = (d_1, \ldots, d_{\|s\|-1}$ be a coordinate in $C_{s,n,\delta}$. Suppose that $d_i \leq n$. Then it is straightforward to identify the image of γ_i in $\mathcal{D}_{s,n,\delta}$. If $d_i \leq n$ for all i then define

$$\phi_{s,\delta}^{\epsilon_0} = \mathfrak{h}_{\gamma_{\|s\|-1}} \circ \cdots \circ \mathfrak{h}_{\gamma_1}.$$

Each $\mathfrak{h}$ map in this section is understood to be composed with a small isotopy. If there is no curve γ_j then $\mathfrak{h}_{\gamma_j}$ is defined to be just a small isotopy.

If $d_i > n+1$ then the image of γ in $\mathcal{D}_{s,n,\delta}$ may require some choices. Suppose that the diagram appears as in the bottom left corner of Figure 15 around the stabilized component. (We may do so without loss of generality: the other possibility is a mirror of this figure.) The n-th canonical one-handle η between $\bar{t}_i$ and t_i intersects γ_i. We may assume that the curves intersect only once and that η is horizontal. If no handles between $\bar{t}_{i+1}$ and t_{i+1} have been attached, then one can slide the feet of γ_i to a new curve γ_i' so that γ_i' and η are disjoint and γ_i' lies above η. It follows that no other canonical one-handle intersects γ_i'. This is the situation in the bottom row of Figure 15: the boxed diagram is part of a duplicated column. The solid arrow is γ.

Let η' be the n-th canonical one-handle between $\bar{t}_{i+1}$ and t_{i+1}. This arc does not intersect γ_i. Nevertheless, if η' is attached before η, it is possible that the feet of γ cannot be slid as above. (For example, they may lie on different components.) To prevent this, slide the feet of γ_i as in the first column of Figure 15. The figure shows that all of this footwork is consistent: the curve γ_i' in the top right corner is the same whichever way one proceeds through the square. Moving forward we will always write γ_i for the image of γ_i or γ_i' as appropriate.

Every axis of $\widetilde{H}_s$ has an extra canonical one-handle attachment. Call the curve of attachment in the i-th column η_i. There is a curve $\eta_i^{\perp}$ which is the image of η_i under $\gamma_i^{\perp}$. (Loosely, $\eta_i^{\perp}$ the image of η "upstairs" in $H_{s,n}$.) Suppose that $\delta = (d_1, \ldots, d_{\|s\|-1})$ is a coordinate with $d_j \geq n+1$. Define...

Write $\mathcal{D}_{(j_1,j_2)}$ and $C_{(j_1,j_2)}$ for the diagram and group at (j_1, j_2) in H_n. Write $\widetilde{\mathcal{D}}_{(j_1,j_2)}$ and $\widetilde{C}_{(j_1,j_2)}$ for the diagram and group at (j_1, j_2) in $\widetilde{H}_n$.

- Suppose that (j_1, j_2) is pre-stabilization. Define $\phi\colon C_{(j_1,j_2)} \to \widetilde{C}_{(j_1,j_2)}$ to be $\mathfrak{h}_\gamma$, the one-handle attachment map along γ along with a small isotopy.
- Consider the vertex $(n+1, j_2)$ with $j_2 < n+1$. Let $\gamma^{\perp}$ be the arc dual to γ in $\widetilde{\mathcal{D}}_{(n+1,j_2)}$ so that $\mathfrak{h}_{\gamma^{\perp}} : \widetilde{\mathcal{D}}_{(n+1,j_2)} \to \mathcal{D}_{(n+1,j_2)}$. There is a canonical surgery arc $\widetilde{\eta}$ so that $\mathfrak{h}_{\widetilde{\eta}}\colon \widetilde{\mathcal{D}}_{(n+1,j_2)} \to \widetilde{\mathcal{D}}_{(n+2,j_2)}$. Write η_0 for the image $\widetilde{\eta}$ after attaching a one-handle along $\gamma^{\perp}$. On any side vertex with $j_1 > n+1$, define $\phi = \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0}$ along with a small isotopy.
- Define η_1 analogously by looking at the vertex $(j_1, n+2)$ with $j_1 < n+1$. On any side vertex with $j_2 > n+1$ define $\phi = \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1}$ along with a small isotopy.

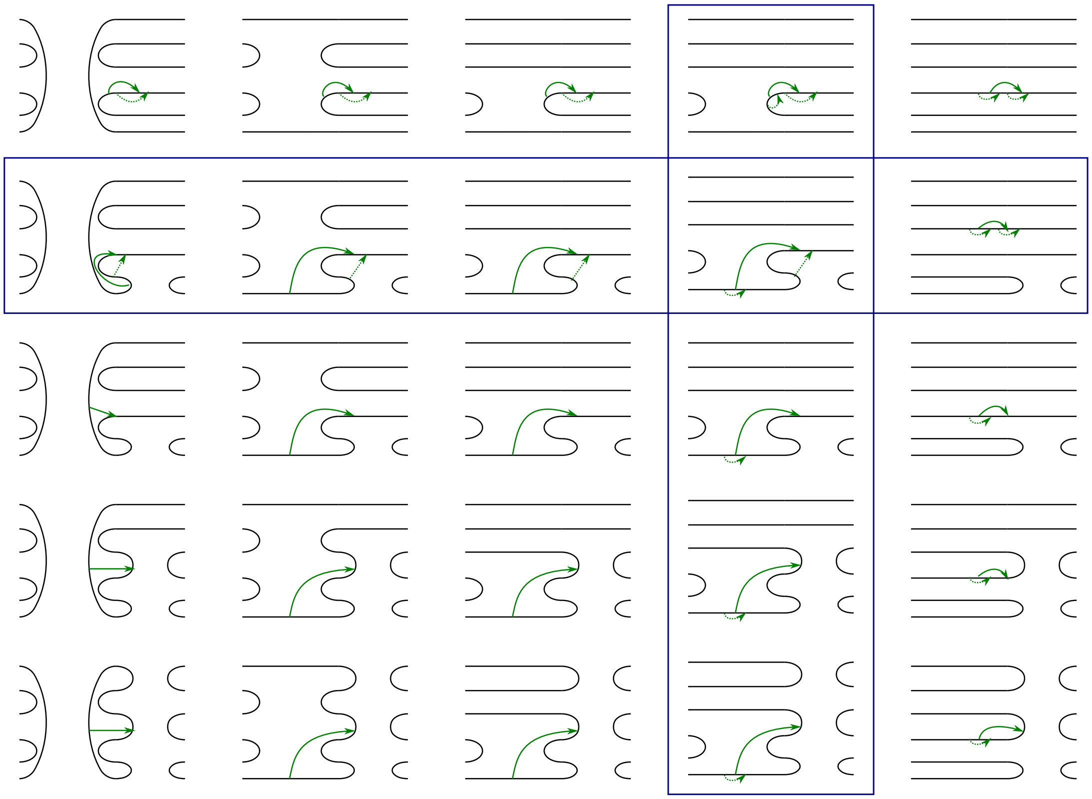

FIGURE 15

- For any post-stabilization vertex define $\phi = \mathfrak{h}_{\gamma} \circ \mathfrak{h}_{\eta_1} \circ \mathfrak{h}_{\eta_0}$ along with a small isotopy.

The situation is shown in Figure 15. The solid arrow on each picture is (the image of) γ. The dotted arrows are η_0 and η_1. The boxed row and column are the duplicates. It is easy to see that ϕ defines a map on any square which lies entirely inside one region.

Let's study the square with lower-left corner (n, n). (The other unsettled squares are simpler.) By construction,

$$\phi^{(0,0)}_{(n,n)} = \mathfrak{h}_{\gamma}, \qquad \phi^{(0,0)}_{(n,n+1)} = \mathfrak{h}_{\gamma} \circ \mathfrak{h}_{\eta_0}$$

$$D^{(0,1)}_{(n,n)} = \mathrm{Id}, \qquad \widetilde{D}^{(0,1)}_{(n,n)} = \mathfrak{h}_{\eta_0}$$

Now set $\phi^{(0,1)}_{(n,n)} = \mathfrak{h}_{\eta_0\gamma}$. (Recall that $\mathfrak{h}_{\eta_0\gamma}$ is the associator between $\mathfrak{h}_{\eta_0} \circ \mathfrak{h}_\gamma$ and $\mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0}$. It counts configurations which use both η_0 and γ.)

$$\begin{aligned}
&\left(\phi^{(0,0)}_{(n,n)} \circ D^{(0,1)}_{(n,n)} + \widetilde{D}^{(0,1)}_{(n,n)} \circ \phi^{(0,0)}_{(n,n+1)}\right) + \left(\phi^{(0,1)}_{(n,n)} \circ D^{(0,0)}_{(n,n)} + \widetilde{D}^{(0,0)}_{(n,n)} \circ \phi^{(0,1)}_{(n,n)}\right) \\
&= \left(\mathfrak{h}_{\eta_0} \circ \mathfrak{h}_\gamma + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0}\right) + \left(\mathfrak{h}_{\gamma\eta_0} \circ \partial + \tilde{\partial} \circ \mathfrak{h}_{\gamma\eta_0}\right) \\
&= 0
\end{aligned}$$

This shows that ϕ, restricted to the left edge of the (n,n) square, is a map. One can do the same for the bottom edge by setting $\phi^{(1,0)}_{(n,n)} = \mathfrak{h}_{\eta_1\gamma}$. For the top edge we have

$$\begin{aligned}
\phi^{(0,0)}_{(n,n+1)} &= \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0}, & \phi^{(0,0)}_{(n+1,n+1)} &= \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1} \circ \mathfrak{h}_{\eta_0} \\
D^{(1,0)}_{(n,n+1)} &= \mathrm{Id}, & \widetilde{D}^{(1,0)}_{(n,n+1)} &= \mathfrak{h}_{\eta_1}.
\end{aligned}$$

Set $\phi^{(1,0)}_{(n,n+1)} = \mathfrak{h}_{\gamma\eta_1} \circ \mathfrak{h}_{\eta_0}$ so that

$$\begin{aligned}
&\left(\phi^{(0,0)}_{(n+1,n+1)} \circ D^{(1,0)}_{(n,n+1)} + \widetilde{D}^{(1,0)}_{(n,n+1)} \circ \phi^{(0,0)}_{(n,n+1)}\right) + \left(\phi^{(1,0)}_{(n,n+1)} \circ D^{(0,0)}_{(n,n+1)} + \widetilde{D}^{(0,0)}_{(n+1,n+1)} \circ \phi^{(1,0)}_{(n,n+1)}\right) \\
&= \left(\mathfrak{h}_{\eta_1} \circ \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0} + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1} \circ \mathfrak{h}_{\eta_0}\right) + \left(\mathfrak{h}_{\gamma\eta_1} \circ \mathfrak{h}_{\eta_0} \circ \partial + \widetilde{\partial} \circ \mathfrak{h}_{\gamma\eta_1} \circ \mathfrak{h}_{\eta_0}\right) \\
&= \left(\mathfrak{h}_{\eta_1} \circ \mathfrak{h}_\gamma + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1}\right) \circ \mathfrak{h}_{\eta_0} + \left(\mathfrak{h}_{\gamma\eta_1} \circ \partial\right) \circ \mathfrak{h}_{\eta_0} + \left(\widetilde{\partial} \circ \mathfrak{h}_{\gamma\eta_1}\right) \circ \mathfrak{h}_{\eta_0} \\
&= 0
\end{aligned}$$

For the right edge we have

$$\begin{aligned}
\phi^{(0,0)}_{(n+1,n)} &= \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1}, & \phi^{(0,0)}_{(n+1,n+1)} &= \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0} \circ \mathfrak{h}_{\eta_1} \\
D^{(0,1)}_{(n+1,n)} &= \mathrm{Id}, & \widetilde{D}^{(0,1)}_{(n+1,n)} &= \mathfrak{h}_{\eta_0}.
\end{aligned}$$

Set $\phi^{(0,1)}_{(n+1,n)} = \mathfrak{h}_{\gamma\eta_0} \circ \mathfrak{h}_{\eta_1} + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0\eta_1}$. Then

$$\begin{aligned}
&\left(\phi^{(0,0)}_{(n+1,n+1)} \circ D^{(0,1)}_{(n+1,n)} + \widetilde{D}^{(0,1)}_{(n+1,n)} \circ \phi^{(0,0)}_{(n+1,n)}\right) + \left(\phi^{(0,1)}_{(n+1,n)} \circ D^{(0,0)}_{(n+1,n)} + \widetilde{D}^{(0,0)}_{(n+1,n+1)} \circ \phi^{(0,1)}_{(n+1,n)}\right) \\
&= \left(\mathfrak{h}_{\eta_0} \circ \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1} + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1} \circ \mathfrak{h}_{\eta_0}\right) + \left(\mathfrak{h}_{\gamma\eta_0} \circ \mathfrak{h}_{\eta_1} \circ \partial + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0\eta_1} \circ \partial + \partial \circ \mathfrak{h}_{\gamma\eta_0} \circ \mathfrak{h}_{\eta_1} + \partial \circ \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0\eta_1} \right. \\
&= \left(\mathfrak{h}_{\eta_0} \circ \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1} + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_1} \circ \mathfrak{h}_{\eta_0}\right) + \left(\mathfrak{h}_{\gamma\eta_0} \circ \partial + \partial \circ \mathfrak{h}_{\gamma\eta_0}\right) \circ \mathfrak{h}_{\eta_1} + \mathfrak{h}_\gamma \circ \left(\mathfrak{h}_{\eta_0\eta_1} \circ \partial + \partial \circ \mathfrak{h}_{\eta_0\eta_1}\right) \\
&= \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0} \circ \mathfrak{h}_{\eta_1} + \mathfrak{h}_\gamma \circ \mathfrak{h}_{\eta_0} \circ \mathfrak{h}_{\eta_1} \\
&= 0.
\end{aligned}$$

The only outstanding map is $\phi^{(1,1)}_{(n,n)}$. To define it, let's look the other terms of $\widetilde{D} \circ \phi + \phi \circ D \colon C_{(n,n)} \to \widetilde{C}_{(n+1,n+1)}$. We can write this as the sum of six terms, one for each of the directions out of the vertex $(n, n, 0)$.

$$\underbrace{\mathfrak{h}_{\gamma\eta_1} \circ \mathfrak{h}_{\eta_0} \circ \mathrm{Id}}_{(0,1,0)} + \underbrace{(\mathfrak{h}_{\gamma\eta_0} \circ \mathfrak{h}_{\eta_1} + \mathfrak{h}_{\gamma} \circ \mathfrak{h}_{\eta_0\eta_1}) \circ \mathrm{Id}}_{(1,0,0)} + \underbrace{\mathfrak{h}_{\eta_0\eta_1} \circ \mathfrak{h}_{\gamma}}_{(0,0,1)}$$
$$+ \underbrace{0}_{(1,1,0)} + \underbrace{\mathfrak{h}_{\eta_1} \circ \mathfrak{h}_{\eta_0\gamma}}_{(0,1,1)} + \underbrace{\mathfrak{h}_{\eta_0} \circ \mathfrak{h}_{\eta_1\gamma}}_{(1,0,1)}$$

These are the six ways to associate γ, η_0, and η_1. So define $\phi^{(1,1)}_{(n,n)} = \mathfrak{h}_{\gamma_0\eta_0\eta_1}$.

This completes the construction of the map $\phi\colon\ H_n \to \widetilde{H}$.

8. Hochschild homology invariants

This section contained the withdrawn claim. The idea is that a certain natural cobordism amounts to a Hochchild cocycle on the A_∞-algebra for a triplane diagram. It is more convenient to compute this cocycle in the "perturbed" algebra $\mathcal{A}_S(\mathbf{t})$ using the Sarkar-Seed-Szabó perturbation of Szabó homology. Then one can pair this cocycle with some natural Hochschild cycle to get a numerical invariant. Unfortunately, what we previously believed to be a cocycle is not.

9. Hyperboxes of chain complexes and A_∞ categories

This section is dedicated to establishing the algebraic framework for our A_∞-categories. This framework extends the constructions of Manolescu and Ozsváth in their mammoth paper on Heegaard Floer homology [15]. We show that from a collection of hyperboxes satisfying some coherence conditions one can build an A_∞-category. Theorem 9.19 states this construction is functorial up to homotopy. Corollary 9.8 and Sections 9.2.2 and 9.3 are new results which we hope will be useful in future work. The construction is extended to A_∞-bimodules in upcoming work [2].

9.1. A_∞-categories.

See [9] for an introduction to A_∞-categories, [12] for an exhaustive resource, and [23] for something in between. We avoid tricky sign conventions by working over a ring R of characteristic two.

Definition. An A_∞-*category* $\mathcal{A}$ over R consists of

- a set of *objects* $\mathrm{Ob}(\mathcal{A})$
- an R-module of *morphisms* $\mathrm{Hom}_{\mathcal{A}}(o, o')$ for each pair of objects $o, o' \in \mathrm{Ob}(\mathcal{A})$.
- for every sequence of objects $(o_0, \ldots, o_n)$ a map

$$\mu_n\colon\ \mathrm{Hom}_{\mathcal{A}}(o_0, o_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(o_{n-1}, o_n) \to \mathrm{Hom}_{\mathcal{A}}(o_0, o_n).$$

These maps satisfy the A_∞-*relations*

$$\sum_{i+j+k=n} \mu_{i+1+k}\left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}\right) = 0. \tag{8}$$

Moreover the morphism sets are graded and μ_n has degree $2-n$.

With $n=1$, equation (8) states that μ_1 is a differential on $\operatorname{Hom}(o,o)$ for any $o \in \operatorname{Ob}(\mathcal{A})$. With $n=2$, it states that μ_2 is a chain map, and that μ_3 is a chain homotopy between $\mu_2 \circ (\mu_2 \otimes \operatorname{Id})$ and $\mu_2 \circ (\operatorname{Id} \otimes \mu_2)$.

Definition. Let $(\mathcal{A},\mu)$ and $(\mathcal{B},\mu')$ be A_∞-categories over R. An *A_∞-functor* f consists of

- a map of sets $f\colon \operatorname{Ob}(\mathcal{A}) \to \operatorname{Ob}(\mathcal{B})$.
- for every sequence of objects $(o_0,\dots,o_n)$ a map
$$f_n\colon \operatorname{Hom}_{\mathcal{A}}(o_0,o_1)\otimes\cdots\otimes \operatorname{Hom}_{\mathcal{A}}(o_{n-1},o_n) \to \operatorname{Hom}_{\mathcal{A}}(f(o_0),f(o_n))$$
of degree $1-n$. These maps satisfy, for each $n \geq 1$,

$$\sum_{i+j+k=n} f_{i+1+j}\left(\operatorname{Id}^{\otimes i}\otimes \mu_j \otimes \operatorname{Id}^{\otimes k}\right) = \sum_{i_1+\cdots+i_r=n} \mu'_r\left(f_{i_1}\otimes\cdots\otimes f_{i_r}\right). \tag{9}$$

The *identity functor* $\operatorname{Id}\colon \mathcal{A}\to\mathcal{A}$ is the functor with $f=\operatorname{Id}$, $f_1=\operatorname{Id}$, and $f_i=0$ for $i>1$.

If $f\colon \mathcal{A}\to\mathcal{B}$ and $g\colon \mathcal{B}\to\mathcal{C}$ are A_∞-functors, their composition $(g\circ f)$ is defined by composing the maps of objects and setting

$$(g\circ f)_n = \sum_{i_1+\cdots+i_r=n} f_r(g_{i_1}\otimes\cdots\otimes g_{i_r}).$$

Definition. Let $f,g\colon \mathcal{A}\to\mathcal{B}$ be A_∞-functors. An *A_∞-chain homotopy* between f and g consists of a collection of multilinear maps

$$h_n\colon \operatorname{Hom}_{\mathcal{A}}(o_0,o_1)\otimes\cdots\otimes \operatorname{Hom}_{\mathcal{A}}(o_{n-1},o_n) \to \operatorname{Hom}(f(o_0),f(o_n))$$

of degree $-n$ so that

$$f_n - g_n = \sum_{i_1+\cdots+i_r+k+j_1+\cdots+j_s=n} \mu'_{r+1+s}(f_{i_1}\otimes\cdots\otimes f_{i_r}\otimes h_k \otimes g_{j_1}\otimes\cdots\otimes g_{j_s}) \tag{10}$$

$$+ \sum_{a+b+c=n} h_{a+1+c}(\operatorname{Id}^{\otimes a}\otimes\mu_b\otimes\operatorname{Id}^{\otimes c}) \tag{11}$$

A functor which is A_∞-chain homotopic to the identity functor is called a *chain homotopy equivalence*.

An A_∞-category is not a category because the binary composition μ_2 is not associative. The *homology* of $\mathcal{A}$, denoted $H^*(\mathcal{A})$ is the category with an object $H^*(\operatorname{Hom}(o,o),\mu_1)$ for each $o\in\operatorname{Ob}(\mathcal{A})$. Equation (**??**) implies that each $f\in\operatorname{Hom}(o,o')$ is a chain map, so define $H^*(\operatorname{Hom}(o,o'))$ to be the set of induced maps on homology. Equation (**??**) implies that $H^*(\mathcal{A})$ is an honest category. An A_∞-functor defines an honest functor between homology categories, and an A_∞-chain homotopy equivalence defines an isomorphism of functors.

9.2. **Hyperboxes of chain complexes.** This section recalls the hyperbox constructions in [15]. Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n_{\geq 0}$. Write $E(\mathbf{d})$ for the box with dimensions $\mathbf{d}$ and its initial corner at the origin. Write $E(n)$ for the n-dimensional box with dimensions $(1, \dots, 1)$, i.e. the *n-dimensional hypercube.*

Definition. An *n-dimensional hyperbox of chain complexes of shape* $\mathbf{d}$ is a collection of graded chain complexes

$$\bigoplus_{\delta \in E(\mathbf{d})} C_\delta$$

and a collection of linear maps

$$D^\epsilon_\delta \colon C^\delta \to C^{\delta+\epsilon}$$

with $\delta \in E(\mathbf{d})$ and $\epsilon \in E_n$ so that:

- If $\delta + \epsilon \notin E(\mathbf{d})$ then $D^\epsilon_\delta = 0$.
- The map $D^{\mathbf{0}}_\delta \colon C^\delta \to C^\delta$ is the differential on C^δ.
- The map D^ϵ_δ has degree $1 - \|\epsilon\|$.[4]
- Each hypercube of any dimension is a chain complex. In other words, fixing any $\epsilon'' \in E(n)$,

$$\sum_{\epsilon+\epsilon'=\epsilon''} D^{\epsilon'}_{\delta+\epsilon} \circ D^\epsilon_\delta = 0 \tag{12}$$

  for all $\delta \in E(\mathbf{d})$.

Informally, a hyperbox is a collection of cubical chain complexes which have been stacked. Note that the last condition does not imply that a hyperbox of chain complexes is itself a chain complex! We write $H = (C, D)$, $C = \bigoplus C_\delta$, and $D = \bigoplus D^\epsilon_\delta$ for a generic hyperbox of chain complexes. Here are some examples.

A 0-dimensional hyperbox of chain complexes is a graded chain complex.

A 1-dimensional hyperbox of chain complexes of size (d) is a collection of chain complexes C_i for $i \in \{0, \dots, d\}$ and chain maps $f_i \colon C_i \to C_{i+1}$. In the above notation, $C = \bigoplus_{i=0}^d C_i$ and D is the direct sum of all the differentials on the C_i and the maps f_i. (Equation 12 implies that the f_i must be chain maps.) A one-dimensional hyperbox can be thought of as the "factored mapping cone" of $f_{d-1} \circ \cdots \circ f_0$.

[4]In [15], D^ϵ_δ is required to have degree $\|\epsilon\| - 1$. One could alternatively refer to our definition as a hyperbox of *co*chain complexes, but this isn't a difference worth emphasizing.

A 2-dimensional hyperbox of chain complexes of size (d_1, d_2) is a collection of chain complexes $\{C^{i,j}\}$ for $0 \leq i \leq d_1$ and $0 \leq j \leq d_2$ along with maps

$$\begin{aligned} f^{(1,0)}_{i,j}&\colon C_{i,j} \to C_{i+1,j} && \text{(horizontal maps)}\\ f^{(0,1)}_{i,j}&\colon C_{i,j} \to C_{i,j+1} && \text{(vertical maps)}\\ f^{(1,1)}_{i,j}&\colon C_{i,j} \to C_{i+1,j+1} && \text{(diagonal maps)}\end{aligned}$$

Equation 12 implies that the horizontal and vertical maps are chain maps and that $f^{(1,1)}_{i,j}$ is a homotopy between $f^{(0,1)}_{i+1,j} \circ f^{(1,0)}_{i,j}$ and $f^{(1,0)}_{i,j+1} \circ f^{(0,1)}_{i,j}$.

An n-dimensional hyper<u>cube</u> of complexes is a cubical chain complex with diagonal maps. Such complexes underlie many spectral sequences in low-dimensional topology.

Our standing notation will be that $\delta \in E(\mathbf{d})$ (a "coordinate vector") and $\epsilon \in E_n$ (a "direction vector"). δ is called a *corner* if every coordinate in δ is either maximal or zero. In other words, there is some $\epsilon \in E_n$ so that

$$\delta = (d_1\epsilon_1, \ldots, d_n\epsilon_n).$$

We call δ the *ϵ-corner* of H. We sloppily append coordinates to a vector by writing e.g. $(\delta, 1)$ for $(\delta_1, \ldots, \delta_n, 1)$.

Definition 9.1. Let ${}^0H$ and ${}^1H$ be hyperboxes of chain complexes. A *map of hyperboxes* $F\colon {}^0H \to {}^1H$ is a hyperbox of size $(\mathbf{d}, 1)$ so that the $d_{n+1} = 0$ face of F is ${}^0H$ and the $d_{n+1} = 1$ face of F is ${}^1H$ with grading shifted up by 1.

A map of hyperboxes is determined by the edges with positive $(n+1)$-st coordinate, i.e. maps

$$F^{\epsilon}_{\delta}\colon {}^0C_{\delta} \to {}^1C_{\delta+\epsilon}$$

of degree $\|\epsilon\|$ satisfying

$$\sum (D^{\epsilon}_{\delta+\epsilon'} \circ F^{\epsilon'}_{\delta} + F^{\epsilon'}_{\delta+\epsilon} \circ D^{\epsilon}_{\delta}) = 0 \tag{13}$$

for all $\epsilon \in E(\mathbf{d}, 0)$, and all ϵ' with $(n+1)$-coordinate 1 so that $\epsilon + \epsilon' \leq \epsilon_1$. Conversely, a collection of maps from C to C' satisfying these relations defines a map of hyperboxes.

Let $F\colon {}^0H \to {}^1H$ and $G\colon {}^1H \to {}^2H$ be maps of hyperboxes. Their composition $G \circ F\colon {}^0H \to {}^2H$ is defined by

$$(G \circ F)^{\epsilon'}_{\epsilon}\colon {}^0C^{\epsilon} \to {}^2C^{\epsilon+\epsilon'}$$

$$(G \circ F)^{\epsilon'}_{\epsilon} = \sum_{\epsilon'' \leq \epsilon'} G^{\epsilon'-\epsilon''}_{\epsilon+\epsilon''} \circ F^{\epsilon}_{\epsilon''} \tag{14}$$

In terms of boxes: glue the $d_{n+1} = 1$ face of F to the $d_{n+1} = 0$ face of G to obtain a hyperbox of shape $(\mathbf{d}, 2)$, then compose all possible combinations of maps in the

d_{n+1} direction. Charmingly, a chain homotopy can be thought of as a map of these hyperboxes:

Definition 9.2. Let $F, G : {}^0H \to {}^1H$. A *chain homotopy* from F to G is a hyperbox J of size $(\mathbf{d}, 1, 1)$ so that

- the $\delta_{n+2} = 0$ face of J is F.
- the $\delta_{n+2} = 1$ face of J is G.
- if $\epsilon = (0, \ldots, 0, 1)$ then $J^\epsilon_\delta = \mathrm{Id}$.
- if $\epsilon = (\epsilon', 0, 1)$ and $\epsilon' \neq 0$ then $J^\epsilon_\delta = 0$.

The last two points can be summarized as "the restriction of J_s to a face with $\delta_{n+1} = 0$ is the identity map."

A map of zero-dimensional hyperboxes is the mapping cone of a chain map.

A map of n-dimensional hypercubes is an $(n+1)$-dimensional hypercube, i.e. the mapping cone of a map of cubical complexes. A chain homotopy of maps of hypercubes is equivalent to a chain homotopy of chain maps of cubical complexes.

9.2.1. *Compression.* There is a recursive recipe called *compression* for building a cubical chain complex from a hyperbox of chain complexes. Let $H = (C, D)$ be a hyperbox of chain complexes of shape $\mathbf{d} = (d_1, \ldots, d_n)$. Let $\widehat{C}$ be the cubical complex whose underlying space is the sum of the corners of H. One can construct a differential $\widehat{D}$ on $\widehat{C}$ from H. The hypercube $\widehat{H} = (\widehat{C}, \widehat{D})$ is the *compression* of H. This recipe was first described by Manolescu and Ozsváth. We take an alternative view due to Liu [13].

Let H be a one-dimensional hyperbox of shape (d). Define

$$\begin{aligned} \widehat{C} &= C_0 \oplus C_d \\ \widehat{D}^1_0 &= f_{d-1} \circ \cdots \circ f_0 \\ \widehat{D}^0_0 &= D^0_0, \quad \widehat{D}^0_1 = D^0_d \end{aligned}$$

and $\widehat{H} = (\widehat{C}, \widehat{D})$.

Let H be an n-dimensional hyperbox with shape $(d_1, \ldots, d_n)$ and $d_n > 1$. We can think of H as d_n hyperboxes of shape $(d_1, \ldots, d_{n-1}, 1)$ attached along faces. Label these hyperboxes as H^n_1, H^n_2, and so on. Each of these boxes is a map of hyperboxes of dimension $n-1$.

Definition 9.3. Define H^n to be the map

$$H^n = H^n_{d_n} \circ \cdots \circ H^n_1.$$

H^n is the *partial compression of H along the n-th axis*, or just the *n-th partial compression.* It has shape $(d_1, \ldots, d_{n-1}, 1)$. If $d_n = 1$, then $H^n = H$.

Definition 9.4. Let H be an n-dimensional hyperbox. Define

$$\widehat{H} = H^{n,n-1,\dots,1}$$

In other words, $\widehat{H}$ is the result of n partial compressions starting with the n-th and ending with first. $\widehat{H}$ is a hypercube of dimension n.

It's a helpful exercise to describe $\widehat{H}$ more explicitly for a two-dimensional hyperbox H. (And these compressions play a starring role in Section 8.) First suppose that H has shape $(d,1)$. Then $H^2 = H$. Think of H as d maps of one-dimensional hyperboxes: to compute $\widehat{H} = H^{2,1}$, compose those maps. So

$$\widehat{D}^{(0,1)}_{(0,0)} = D^{(0,1)}_{(0,0)}$$
$$\widehat{D}^{(1,0)}_{(0,0)} = D^{(1,0)}_{(d-1,0)} \circ \cdots \circ D^{(1,0)}_{(1,0)} \circ D^{(1,0)}_{(0,0)}$$
$$\widehat{D}^{(1,0)}_{(0,1)} = D^{(1,0)}_{(d-1,1)} \circ \cdots \circ D^{(1,0)}_{(1,1)} \circ D^{(1,0)}_{(0,1)}$$

$\widehat{D}^{(1,1)}_{(0,0)}$ is a sum of maps, one for each path of a certain type in H from the initial vertex to the terminal vertex. Such a path must include exactly one diagonal edge – in fact, it's totally determined by that edge. We can describe $\widehat{D}^{(1,1)}_{(0,0)}$ by the schematic in Figure 16. (The thick blue diagonal represents a step which appears exactly once while the black lines are of any non-negative length.)

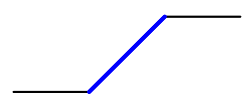

FIGURE 16

Now suppose instead that H has shape (d_1, d_2). Then H^2 is a hyperbox of shape $(d_1, 1)$. Each square in H^2 is the compression of a hyperbox of size $(1, d_2)$. Therefore their diagonal maps are given by Figure 17. Now we can apply the argument for hyperboxes of shape $(d,1)$ inserting Figure 17 for the diagonal.

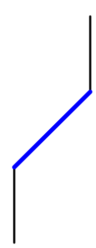

FIGURE 17

The result is that $\widehat{H}$ has underlying space

$$C_{(0,0)} \oplus C_{(1,0)} \oplus C_{(0,1)} \oplus C_{(1,1)}.$$

The vertical (resp. horizontal) maps are compositions of vertical (resp. horizontal) maps in H. The diagonal map is a sum of maps, one for each diagonal edge in H. This diagonal completely describes a path from $(0,0)$ to (d_1, d_2) of the shape in Figure 18. Readers familiar with hyperboxes will recognize that this description of $\widehat{H}$ agrees with Manolescu and Ozsváth's.

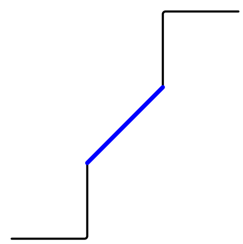

FIGURE 18

Proposition 9.5 ([13]). *This definition of compression agrees with Manolescu and Ozsváth's.*

If $G\colon H \to H'$ is a map of hyperboxes, then $\widehat{G}\colon \widehat{H} \to \widehat{H'}$. Suppose that $F\colon H' \to H''$ is another map of hyperboxes. It would be nice if

$$\widehat{F \circ G} = \widehat{F} \circ \widehat{G}.$$

But this formula is false. Write FG for the hyperbox given by gluing together F and G along the appropriate faces so that $(FG)^{n+1} = F \circ G$ and therefore $\widehat{FG} = \widehat{F \circ G}$. On the other hand, $\widehat{F} \circ \widehat{G}$ is computed by first fully compressing both F and G. In summary,

$$\widehat{F} \circ \widehat{G} = (FG)^{n,\ldots,1,n+1}$$

$$\widehat{F \circ G} = (FG)^{n+1,n,\ldots,1}.$$

Nevertheless,

Lemma 9.6. *$\widehat{F \circ G} \simeq \widehat{F} \circ \widehat{G}$.*

Proof. We work by induction on the dimension of H. Suppose that H, H', and H'' are n-dimensional with $n > 1$ and assume that the theorem holds for $(n-1)$-dimensional hyperboxes. We can think of H, H', and H'' as one-dimensional hyperboxes *in the category of hyperboxes* – a hyperhyperbox. At each vertex is an

$(n-1)$-dimensional hyperbox of chain complexes. In this language FG is a two-dimensional hyperhyperbox. The two ways to compress this hyperhyperbox yield $FG^{n+1,n}$ and $FG^{n,n+1}$. By hypothesis, these two are chain homotopy equivalent.

Consider FG^n as a two-dimensional hyperhyperbox of size $(d_{n-1}, 2)$. The vertices are compressed along the n-th axis and the maps are adjusted accordingly. Repeating the argument above, $FG^{n,n+1,n-1}$ is chain homotopy equivalent to $FG^{n,n-1,n+1}$. Continue $n-2$ more times to prove the theorem.

Now we prove the base case: if FG is a hyperbox of size $(d, 2)$ then $FG^{2,1} \simeq FG^{1,2}$. If $d = 1$ there is nothing to show, so assume $d > 1$. The vertical and horizontal maps agree, so we only need to study the two diagonal maps. Call them p and q where p follows the scheme in Figure 18 and q follows the mirror. Both p and q are sums of maps along certain paths in the cube, one for each diagonal. Write $p_{i,j}$ and $q_{i,j}$ for the terms in $p_{i,j}$ and $q_{i,j}$ which use the diagonal from the vertex (i, j). There is a unique, increasing path from $(0, 0)$ to $(d, 2)$ which uses the diagonals at both $(i, 0)$ and $(j, 1)$. Call the linear map obtained by composing maps along this path $h_{i,j}$. Define

$$h = \sum_{i<j} h_{i,j}$$

so that

$$\begin{aligned}
h_{i,j} \circ D^{(0,0)}_{(0,0)} + D^{(0,0)}_{(d,2)} \circ h_{i,j} =& f^{(1,0)}_{(d-1,2)} \circ \cdots \circ f^{(1,0)}_{(j+1,2)} \\
&\circ \left(f^{(1,0)}_{(j,2)} \circ f^{(0,1)}_{(j,1)} + f^{(0,1)}_{(j+1,1)} \circ f^{(1,0)}_{(j,1)} \right) \\
&\circ f^{(1,0)}_{(j-1,1)} \circ \cdots \circ f^{(1,0)}_{(i+1,1)} \\
&\circ f^{(1,1)}_{(i,0)} \\
&\circ f^{(1,0)}_{(i-1,0)} \circ \cdots \circ f^{(1,0)}_{(1,0)} \circ f^{(1,0)}_{(0,0)} \\
&+ \\
& f^{(1,0)}_{(d-1,2)} \circ \cdots \circ f^{(1,0)}_{(j+1,2)} \\
&\circ f^{(1,1)}_{(j,1)} \\
&\circ f^{(1,0)}_{(j-1,1)} \circ \cdots \circ f^{(1,0)}_{(i+1,1)} \\
&\circ \left(f^{(1,0)}_{(i,2)} \circ f^{(0,1)}_{(i,1)} + f^{(0,1)}_{(i+1,1)} \circ f^{(1,0)}_{(i,1)} \right) \\
&\circ f^{(1,0)}_{(i-1,0)} \circ \cdots \circ f^{(1,0)}_{(1,0)} \circ f^{(1,0)}_{(0,0)}
\end{aligned}$$

This equation is easiest to understand in the visual calculus of Figure 19. The thickened, blue edges represent directions which can only be used once. The dot represents the internal differential. (In the algebra of songs this is the result of playing

{}.) From there one can check that

$$\sum_{i<j} h_{i,j} \circ D^{(0,0)}_{(0,0)} + D^{(0,0)}_{(d,2)} \circ h_{i,j} = p + q \tag{15}$$

as in the proof of Stokes' theorem from multi-variable calculus. □

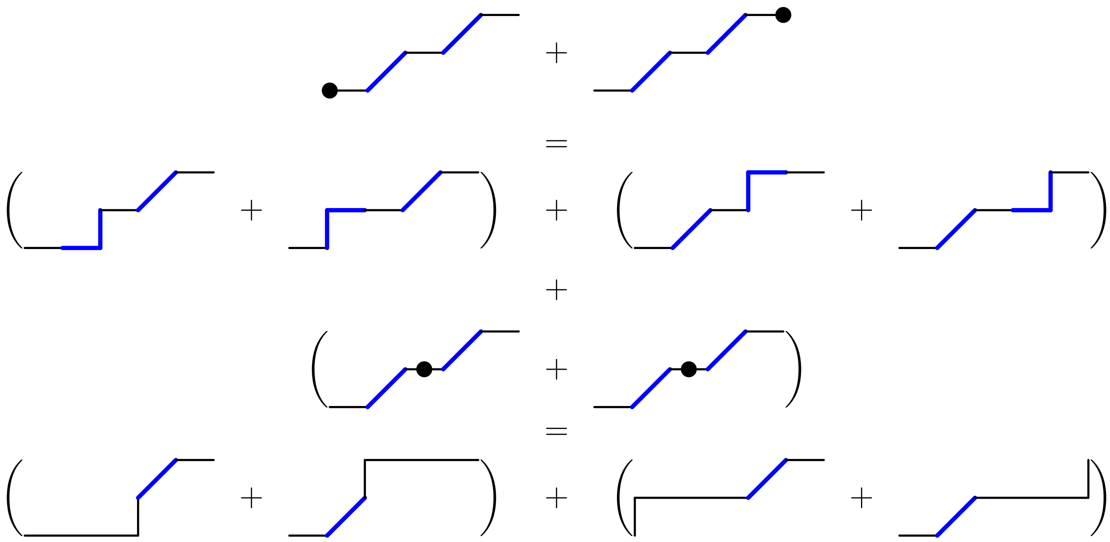

FIGURE 19. A graphical representation of equation (15). The dots represent application of the internal differential; in the musical vocabulary, they are the result of playing {}.

Lemma 9.7. *Suppose that $F \simeq G$ are maps of hyperboxes. Then $\widehat{F} \simeq \widehat{G}$.*

Proof. Let $J\colon F \to G$ be the homotopy. Then $\widehat{J}$ is the mapping cone of a chain map $\mathrm{Id} + j\colon \widehat{F} \to \widehat{G}$. Meanwhile $\widehat{F}$ and $\widehat{G}$ are mapping cones of chain maps f and g. The square of the differential on $\widehat{J}$ is

$$\mathrm{Id} \circ f + g \circ \mathrm{Id} + j \circ D + D \circ j.$$

□

Corollary 9.8. *Compression of hyperboxes is a functor from the homotopy category of hyperboxes of chain complexes to the homotopy category of chain complexes.*

9.2.2. *Two tensor products.* For plain old chain maps f and g,

$$\mathrm{cone}(f \otimes g) \neq \mathrm{cone}(f) \otimes \mathrm{cone}(g).$$

Suppose that $\otimes$ is some kind of tensor product operation on hyperboxes. If $F\colon H_0 \to H_0'$ and $G\colon H_1 \to H_1'$ are maps of hyperboxes, then $F \otimes G$ is analogous to $\mathrm{cone}(f) \otimes \mathrm{cone}(G)$ rather than $\mathrm{cone}(f \otimes g)$. Therefore there are two different tensor products: $\otimes$ for hyperboxes in general and a separate operation, $\boxtimes$, for maps.

Let H be a hyperbox of chain complexes of dimension n and shape $\mathbf{d}$. Let H' be a hyperbox of chain complexes of dimension n' and shape $\mathbf{d}'$. Define $H \otimes H'$ to be the hyperbox of dimension $n + n'$ and shape $(\mathbf{d}, \mathbf{d}')$ whose underlying space is

$$(C \otimes C')_{(\delta,\delta')} = C_\delta \otimes C'_{\delta'}.$$

and whose maps $D^\otimes$ are defined as follows:

$$D^{\otimes,(\epsilon,\epsilon')}_{(\delta,\delta')} = \begin{cases} D^\epsilon_\delta \otimes \mathrm{Id}_{H'_{\delta'}} & \epsilon' = (0,\dots,0) \\ \mathrm{Id}_{H_\delta} \otimes D'^{\epsilon'}_{\delta'} & \epsilon = (0,\dots,0) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 9.9. *$H \otimes H'$ is a hyperbox.*

Proof. Each cube of $H \otimes H'$ is the ordinary tensor product of cubical complexes. □

Lemma 9.10:

(1) $H \otimes (H' \otimes H'') = (H \otimes H') \otimes H''$.

(2) $\widehat{H \otimes H'} = \widehat{H} \otimes \widehat{H'}$. *Equivalently, the differential on* $\widehat{H \otimes H'}$ *has form* $\widehat{D} \otimes \mathrm{Id}_{\widehat{H'}} + \mathrm{Id}_{\widehat{H}} \otimes \widehat{D'}$.

Proof. The first assertion holds because it holds for ordinary cubical chain complexes.

The underlying groups of $\widehat{H \otimes H'}$ and $\widehat{H} \otimes \widehat{H'}$ are identical. The differential on $H \otimes H'$ can be written

$$D^\otimes = \mathrm{Id} \otimes D' + D \otimes \mathrm{Id}$$

where D is the differential on C and D' is the differential on C'. We claim this holds for $(H \otimes H')^{m+n,\dots,k}$ with $0 \le k \le m+n$. Suppose it holds for $k = i$. To construct the next partial compression, one thinks of the boxes in the $(i-1)$-st direction as maps of hyperboxes and composes them. This direction belongs to either H or H'. By hypothesis, all of the maps have the form $\mathrm{Id}_H \otimes g$ or $f \otimes \mathrm{Id}_{H'}$. It follows that the differential on the fully compressed hyperbox has form $\mathrm{Id} \otimes D' + D \otimes \mathrm{Id}$. □

Any chain complex (C', d) can be thought of as a zero-dimensional hyperbox. Therefore $H \otimes C'$ is a hyperbox with the same shape as H and

$$(C \otimes C')_\delta = C_\delta \otimes C'$$
$$D_\delta^{\otimes,\epsilon} = \begin{cases} D_\delta^0 \otimes d & \epsilon = \epsilon_0 \\ D_\delta^\epsilon \otimes \mathrm{Id} & \text{otherwise} \end{cases}$$

Now we define the tensor product of maps of hyperboxes. Let $F\colon H_0 \to H_0'$ and $G\colon H_1 \to H_1'$ be maps of hyperboxes with shapes $(\mathbf{d}, 1)$ and $(\mathbf{d}', 1)$, respectively. Define $F \boxtimes G$ to be the map $H_0 \otimes H_1 \to H_0' \otimes H_1'$ defined by

$$(F \boxtimes G)_{(\epsilon_2,\epsilon_3)}^{(\epsilon,\epsilon')} = F_{\epsilon_2}^\epsilon \otimes G_{\epsilon_3}^{\epsilon'}.$$

Lemma 9.11:

(1) If F and G are maps of cubical complexes, then $F \boxtimes G$ is the usual tensor product of chain maps.
(2) $F \boxtimes G$ really is a map of hyperboxes $\widehat{H_0} \otimes \widehat{H_1}$ to $\widehat{H_0'} \otimes \widehat{H_1'}$.
(3) $(F \boxtimes G) \circ (F' \boxtimes G') = (F \circ F') \boxtimes (G \circ G')$.
(4) $\mathrm{Id} \boxtimes \mathrm{Id} = \mathrm{Id}$.
(5) As a map of chain complexes, $\widehat{F \boxtimes G} = \widehat{F} \otimes \widehat{G}$.

Proof. The first and second statements are straightforward. The third and fourth follow from thinking cube-by-cube. The fifth follows from the second point of Lemma 9.10. The sixth is essentially a consequence of the fact that

$$(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')$$

for linear maps. □

9.3. **From hyperboxes to A_∞-algebras.** First, some notation for sequences and subsequences. Let $\boldsymbol{s} = (s_0, \ldots, s_{k+1})$ be a sequence of natural numbers. Write $|\boldsymbol{s}|$ for the length of $\boldsymbol{s}$. Let $\{C_{ij} : i, j \in \mathbb{N}\}$ be a collection of chain complexes. Set

$$C_{\boldsymbol{s}} = C_{s_0 s_1} \otimes C_{s_1 s_2} \otimes \cdots \otimes C_{s_k s_{k+1}}.$$

For a sequence $\boldsymbol{s} = (s_0)$ of length one, define $C_{\boldsymbol{s}} = C_{s_0 s_0}$.

Write $\boldsymbol{s}' \subset \boldsymbol{s}$ if $\boldsymbol{s}'$ is a sequence, $s_0, s_{k+1} \in \boldsymbol{s}'$, and there is an order-preserving injection $\boldsymbol{s}' \hookrightarrow \boldsymbol{s}$. For a 0-1 sequence ϵ, write $\boldsymbol{s}(\epsilon)$ for the subsequence of $\boldsymbol{s}$ which contains s_i precisely if $\epsilon_i = 0$. For example,

$$\boldsymbol{s}((0, \ldots, 0)) = \boldsymbol{s}$$
$$\boldsymbol{s}((1, \ldots, 1)) = (s_0, s_k + 1).$$

Let ϵ and ϵ' be two 0-1 sequences so that $\epsilon < \epsilon'$. Consider each maximal contiguous subsequence of 0s in ϵ which do not appear in ϵ'. For example, in

$$\begin{aligned}\epsilon &= (\underline{0}, 1, \underline{0}, 0, 1, 0, 1, 0, \underline{0, 0})\\ \epsilon' &= (1, 1, 1, 0, 1, 0, 1, 0, 1, 1)\end{aligned}$$

the maximal subsequences are underlined. Let $c(\epsilon, \epsilon')$ be the set which contains, for each underlined sequence, all the corresponding elements of $\boldsymbol{s}$ and the surrounding ones. For example,

$$\begin{aligned}\boldsymbol{s} &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)\\ \boldsymbol{s}(\epsilon) &= (1, \underline{2, 4}, 5, 7, 9, \underline{10, 11}, 12)\\ \boldsymbol{s}(\epsilon') &= (1, 5, 7, 9, 12)\\ c(\epsilon, \epsilon') &= ((1, 2, 4, 5), (9, 10, 11, 12))\end{aligned}$$

We call $c(\epsilon, \epsilon')$ the *contraction sequence of ϵ and ϵ'*. The *fixed sequence $f(\epsilon, \epsilon')$ of ϵ and ϵ'* is a sequence of pairs of elements from $\boldsymbol{s}(\epsilon)$. Its elements are contiguous pairs of elements of $\boldsymbol{s}(\epsilon)$ which do not appear in $c(\epsilon, \epsilon')$. So in the running example,

$$f(\epsilon, \epsilon') = ((5, 7), (7, 9)).$$

The key point is that the contraction and fixed sequence describe a decomposition of $C_{\boldsymbol{s}}$:

$$C_{\boldsymbol{s}} = \left(\bigotimes_{c' \in c(\epsilon, \epsilon')} C_{c'} \right) \otimes \left(\bigotimes_{f' \in f(\epsilon, \epsilon')} C_{f'} \right).$$

Note that each $C_{f'}$ has a single tensor factor while $C_{c'}$ is a product of many basic factors.

9.3.1. *Systems of hyperboxes.*

Definition 9.12. Let $C = \{C_{i,j}\}$ be a collection of chain complexes indexed by $\mathbb{N} \times \mathbb{N}$. A *system of hyperboxes over C* assigns to each sequence $\boldsymbol{s}$ with $|\boldsymbol{s}| = k$ a $(k-1)$-dimensional hyperbox $H_{\boldsymbol{s}}$ satisfying the following properties:

- The ϵ-corner of $H_{\boldsymbol{s}}$ is $C_{\boldsymbol{s}(\epsilon)}$. (So the initial corner of $H_{\boldsymbol{s}}$ is $C_{\boldsymbol{s}}$ and the terminal corner is $C_{s_0 s_k}$.)
- Let F be the face of $H_{\boldsymbol{s}}$ between the ϵ- and ϵ'-corners. Then

$$F \cong \left(\bigotimes_{c' \in c(\epsilon, \epsilon')} H_{c'} \right) \otimes \left(\bigotimes_{f' \in f(\epsilon, \epsilon')} C_{f'} \right)$$

  where the isomorphism only permutes tensor factors.

The second condition is called the *face condition*. It is a locality condition for the component maps – maps which combine one part of a tensor product should not affect the other parts.

From a system of hyperboxes H one can build an A_∞-category $\mathcal{A}(H)$ as follows. Set $\mathrm{Ob}(\mathcal{A}(H)) = \mathbb{N}$ and $\mathrm{Hom}_{\mathcal{A}(H)}(i,j) = C_{ij}$. Let $\mu_k\colon C_{\boldsymbol{s}} \to C_{s_0 s_{k+1}}$ be the longest diagonal map in $\widehat{H_x}$.

Proposition 9.13. *$\mathcal{A}(H)$ is an A_∞-category.*

Proof. Each diagonal from the origin of $\widehat{H_x}$ corresponds to a 0-1 sequence ϵ and therefore to a contraction sequence $c(\epsilon, \epsilon')$. If $c(\epsilon, \epsilon')$ has more than one element then the corresponding diagonal map vanishes because of the face condition and Lemma 9.10. Therefore the differential on $\widehat{H_x}$, applied to $C_{\boldsymbol{s}}$, is equal to

$$\sum_{i+j+\ell=k} \mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes \ell}.$$

By the face condition, the component of the differential from $C_{\boldsymbol{s}(\epsilon)}$ to $C_{\boldsymbol{s}(1,\dots,1)}$ is $\mu_k \otimes \mathrm{Id}$. The component of $\widehat{D}^2$ which extends from the $(0,\dots,0)$-corner to the $(1,\dots,1)$-corner of $\widehat{H_x}$ is

$$\sum_{i+j+\ell=1} \mu_{i+\ell+1} \circ \left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes \ell}\right) = 0.$$

This is precisely equation (8).

μ_i is computed from the compression of an $(i-1)$-dimensional hyperbox. The compression of a hyperbox is a hyperbox, so it's diagonal has degree $1-i$. This is precisely the degree required in the definition of an A_∞-category. □

This construction constitutes a functor between the homotopy category of systems of hyperboxes of chain complexes and the homotopy category of A_∞-categories.[5] The latter is the A_∞-category category whose objects are A_∞-categories and whose morphisms are A_∞-functors, identifying A_∞-chain homotopic functors.[6] Let's construct the former category.

Definition 9.14. Let $\mathcal{H}$ and $\mathcal{H}'$ be systems of hyperboxes over C and C' so that $H_{\boldsymbol{s}}$ and $H'_{\boldsymbol{s}}$ have the same shape for each $\boldsymbol{s}$. A *map of systems of hyperboxes* is a

[5] A_∞-functors and maps of hyperboxes both compose associatively only up to homotopy, so they do not form honest categories.

[6] This naïve definition agrees with the naïve definition of the homotopy category of A_∞-algebras by restricting to categories with a single object. There is a substantial effort put into verifying that these naïve definitions agree with more sophisticated homotopy-theoretic definitions, but we ignore that here.

collection of maps of hyperboxes

$$G_{\boldsymbol{s}}\colon\ H_{\boldsymbol{s}} \to H'_{\boldsymbol{s}}$$

which satisfies the following face condition. Let F be the (ϵ, ϵ')-face of $H_{\boldsymbol{s}}$. Then $G_{\boldsymbol{s}}$ must satisfy

$$G_{\boldsymbol{s}}|_F = \left(\boxtimes_{s' \in c(\epsilon,\epsilon')} G_{s'} \right) \boxtimes \left(\boxtimes_{f' \in f(\epsilon,\epsilon')} G_{f'} \right)$$

Note that for $f' \in f(\epsilon, \epsilon')$, $G_{f'}\colon\ C_{f'} \to C'_{f'}$ is an ordinary chain map (or more precisely, its mapping cone). Loosely, $G_{\boldsymbol{s}}$ acts on the fixed part of $H_{\boldsymbol{s}}$ by ordinary chain maps.

For any $\mathcal{H}$ there is an identity map $\mathrm{Id}\colon\ \mathcal{H} \to \mathcal{H}$ where $G_{\boldsymbol{s}}$ is the identity map for all $\boldsymbol{s}$.

Definition 9.15. Let $G\colon\ \mathcal{H} \to \mathcal{H}'$ and $G'\colon\ \mathcal{H}' \to \mathcal{H}''$ be maps of systems of hyperboxes. Define

$$(G' \circ G)_{\boldsymbol{s}} = G'_{\boldsymbol{s}} \circ G_{\boldsymbol{s}}.$$

Lemma 9.16. *Definition 9.15 actually defines a map of systems*

$$(G' \circ G)\colon\ \mathcal{H} \to \mathcal{H}''$$

and $(\mathrm{Id} \circ G) = G$ *and* $(G \circ \mathrm{Id}) = G$.

Proof. Follows directly from Lemma 9.11. ☐

A map of systems $\mathfrak{g}\colon\ \mathcal{H} \to \mathcal{H}'$ induces an A_∞-functor $g\colon\ \mathcal{A}(\mathcal{H}) \to \mathcal{A}(\mathcal{H}')$ in the following way. Let $x \in C_{\boldsymbol{s}}$ be a simple tensor of length n. There is a corresponding map of hyperboxes $G_x\colon\ H_x \to H'_x$. Define $g_n(x)$ to be the image of x under the longest diagonal map on $\widehat{G_x}$.

Lemma 9.17. *The identity map on* $\mathbb{N}$ *and* $\{g_n\}_{n=1}^\infty$ *defines an* A_∞*-functor.*

Proof. We aim to show that $\{g_n\}_{n=1}^\infty$ satisfies Equation 9. Consider the differential on $\widehat{G_x}$ where x is a simple tensor of length n. Its restriction to the initial vertex of $\widehat{G_x}$ can be written as

$$\sum_{i+j+\ell=n} \mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes \ell} + \sum_{i_1+\cdots+i_r=n} g_{i_1} \otimes \cdots \otimes g_{i_r}$$

The first term is the part of $\widehat{G_x}$ corresponding to diagonals which do not change the $(n+1)$-st coordinate. The second term corresponds to diagonals which do change that coordinate; it has that form by Lemma 9.11, part (5). Therefore the component of the square of the differential on $\widehat{G_x}$ from the initial vertex to the terminal vertex is

$$\sum_{i+j+\ell=n} g_{j+1} \circ (\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes \ell}) + \mu_r \left(\sum_{i_1+\cdots+i_r=n} g_{i_1} \otimes \cdots \otimes g_{i_r} \right) = 0.$$

□

Definition 9.18. A *chain homotopy* between maps of systems $G, G' \colon \mathcal{H} \to \mathcal{H}'$ is a collection of hyperboxes

$$J_{\boldsymbol{s}} \colon G_{\boldsymbol{s}} \to G'_{\boldsymbol{s}}$$

whose length one maps are identity maps and which also satisfies the following face condition. Let F be a face of $J_{\boldsymbol{s}}$ in which the last component changes. Suppose that the $d_{|\boldsymbol{s}|+1} = d_{|\boldsymbol{s}|+2} = 0$ face of F is the (ϵ, ϵ') face of $H_{\boldsymbol{s}}$. Then

(16)

$$J_{\boldsymbol{s}}|_F = \left(\mathop{\boxtimes}_{\substack{s' \in c(\epsilon,\epsilon') \\ s' = (s''_1, \dots, s_r)}} \bigoplus_{i=1}^{r} \left(F_{s''_1} \boxtimes \cdots \boxtimes F_{s''_{i-1}} \boxtimes J_{s''_i} \boxtimes G_{s''_{i+1}} \boxtimes \cdots \boxtimes G_{s''_r} \right) \right) \otimes \left(\bigotimes_{f' \in f(\epsilon,\epsilon')} \mathrm{Id}_f \right).$$

This definition is rigged to ensure the following theorem.

Theorem 9.19. *Proposition 9.13 and the construction above define a functor from the homotopy category of systems of hyperboxes to the homotopy category of A_∞-categories.*

Proof. Suppose that F and G are chain homotopic maps of systems of hyperboxes. Let $J_{\boldsymbol{s}}$ be the homotopy between $F_{\boldsymbol{s}}$ and $G_{\boldsymbol{s}}$. Let $x \in C_{\boldsymbol{s}}$ be an simple tensor. Define $j_n(x)$ to be the longest diagonal map on $\widehat{J}_x$ applied to x. These maps satisfy equation (10) by the same arguments as above: consider the differential from the initial vertex of $\widehat{J}_s$. It has three types of terms: the identity, application of f_n, the differential on C_s, and diagonals which change both the $(|s|+1)$-st and $(|s|+2)$-nd coordinates. (Any other diagonal map is zero because J_s is a homotopy). Now consider the component of the square of the differential from the initial to the final vertex of $\widehat{J}_s$. The face conditions imply that this has form $\mathrm{Id} \circ f_n$, $g_n \circ \mathrm{Id}^{\otimes |s|}$, $j_n(\mathrm{Id}^{\otimes a} \otimes \mu_i \otimes \mathrm{Id}^{\otimes b})$ and $\mu_i(f_{i_i} \otimes \cdots \otimes f_{i_r} \otimes j_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_r})$. Therefore these maps satisfy equation (10).

Let G and G' be maps of systems so that $\mathfrak{g}' \circ \mathfrak{g}$ is defined. The A_∞-functor induced by $G' \circ G$ is A_∞-chain homotopic to $g' \circ g$ by Corollary 9.8, Definition 9.14, and Lemma 9.11, part 5.

It is straightforward to check that the identity map of systems induces the identity functor. □

Remark 9.20. This section discussed systems of hyperboxes in which summands are indexed by pairs of natural numbers, but of course one could repeat the construction over any set. For example, in Section 3 we work with the set $\{1, 2, 3\}$.

Remark 9.21. We could extend our multiplication maps to $\left(\bigoplus_{i,j} C_{i,j}\right)^{\otimes n}$ by declaring that, for example, if $x \in C_{1,2} \otimes C_{5,9}$ then $\mu_2(x) = 0$. With this convention our construction amounts to a functor from the homotopy category of systems of hyperboxes to the homotopy category of A_∞-algebras.

9.4. **Internal homotopy and enlargements.** There is another way in which two systems of hyperboxes could yield homotopy equivalent A_∞-category. Here is what we have in mind: suppose that $f \simeq g$. It may hold that $f = f_1 \circ f_0$ and $g = g_1 \circ g_0$ but f_0 and g_0 are not chain homotopic. (For example, they may have different codomains.) So the compressions of the hyperboxes $C_0 \underset{f_0}{\longrightarrow} C_1 \underset{f_1}{\longrightarrow} C_2$ and $C_0 \underset{g_0}{\longrightarrow} C_1' \underset{g_1}{\longrightarrow} C_2$ are chain homotopic even if the hyperboxes themselves are not.

Definition 9.22. Let $\mathcal{H}$ and $\mathcal{H}'$ be systems of hyperboxes over C so that $C_{\boldsymbol{s}} \cong C'_{\boldsymbol{s}}$ for all $\boldsymbol{s}$. Suppose that these is an integer ℓ so that

- If $\boldsymbol{s}$ does not contain ℓ, then $H_{\boldsymbol{s}} \cong H'_{\boldsymbol{s}}$ by a map of hyperboxes which extends the isomorphisms $C_{\boldsymbol{s}} \cong C'_{\boldsymbol{s'}}$.
- If $\boldsymbol{s}$ contains ℓ, then think of $H_{\boldsymbol{s}}$ and $H'_{\boldsymbol{s}}$ as a sequence of maps glued together along the dimension containing ℓ. The composition of these maps is chain homotopic by a homotopy which vanishes on any tensor product whose underlying sequence does not contain ℓ.

We say that $\mathcal{H}$ and $\mathcal{H}'$ are *internally chain homotopic.*

Proposition 9.23. *If $\mathcal{H}$ and $\mathcal{H}'$ are internally homotopic and all the chain complexes in both systems are finite-dimensional over R, then $\mathcal{A}(\mathcal{H}) \simeq \mathcal{A}(\mathcal{H}')$.*

Proof. We will define a functor $f\colon \mathcal{A}(\mathcal{H}) \to \mathcal{A}(\mathcal{H}')$. Let x be a simple tensor of length k whose sequence $\boldsymbol{s}$ does not contain ℓ. Define $f_1(x)$ to be the isomorphism in the first sentence.

Suppose that the last occurrence of ℓ in $\boldsymbol{s}$ is at the i-th entry. The key observation is that the partial compressions $H_{\boldsymbol{s}}^{k,\dots,i+1}$ and $H_{s}'^{k,\dots,i+1}$ are homotopic as maps of hyperboxes. This follows from the functoriality of compression. So there is a map

$$F_{\boldsymbol{s}}\colon H_{\boldsymbol{s}}^{k,\dots,i+1} \to H_{\boldsymbol{s}}'^{k,\dots,i+1}$$

whose length one edges are isomorphisms. With the assumption that all the chain complexes are finite-dimensional, a standard argument shows that F_s has an inverse G_s up to homotopy. It follows that

$$H_s^{k,\dots,i+1} \simeq H_s'^{k,\dots,i+1}$$

as hyperboxes. Let f_s be the longest diagonal map in $\widehat{F_s}$. By our previous arguments, the sum of all the f_s defines a functor f. Let g be the reverse functor defined by hyperboxes G_s. We have $G_s \circ F_s \simeq \mathrm{Id}$. It follows that $g \circ f \simeq \mathrm{Id}$. $\square$

Here is a definition from [15] which inspired the proposition above.

Definition 9.24. Let $H = (C, D)$ be a hyperbox of chain complexes of shape $\mathbf{d}$ and dimension n. Fix $k \in \{1, \dots, n\}$ and $j \in \{0, \dots, d_k\}$. Set $\tau_k = (0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in the kth position. Define $\mathbf{d}^+$ by

$$d_i^+ = \begin{cases} d_i & i \neq k \\ d_i + 1 & i = k. \end{cases}$$

Define $H^+ = (C^+, D^+)$ to be the hyperbox of shape $\mathbf{d} + \tau_k$ with

$$C^{+,\epsilon} = \begin{cases} C^\delta & \delta_k \leq j \\ C^{\delta - \tau_k} & \delta_k \geq j + 1 \end{cases}$$

and

$$D_\delta^\epsilon = \begin{cases} D_\delta^\epsilon & \delta_k + \epsilon_k \leq j \\ \mathrm{Id} & \delta_k = j, \epsilon = \tau_k \\ 0 & \delta_k = j, \epsilon_k = 1, \|\epsilon\| > 1 \\ D_{\delta - \tau_k}^\epsilon & \delta > 0. \end{cases}$$

We say that H^+ is the *elementary enlargement of H at k along the j-th axis*.

It is straightforward to show that $\widehat{H^+} \cong \widehat{H}$. (It also follows from Proposition 9.23.)

References

[1] J. A. Baldwin, M. Hedden, and A. Lobb. On the functoriality of Khovanov-Floer theories. *ArXiv e-prints*, September 2015.

[2] John A. Baldwin, Adam Saltz, and Cotton Seed. Szabó's link homology theory for tangles. 2018. Draft available at http://www.adamsaltz.com/images/szabotanglesdraft.pdf.

[3] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geometry & Topology*, 9(3):1443–1499, 2005.

[4] Joan S. Birman. On the stable equivalence of plat representations of knots and links. *Canad. J. Math.*, 28:264–290, 1976.

[5] Tara E. Brendle and Allen Hatcher. Configuration spaces of rings and wickets. *Comment. Math. Helv.*, 88:131–162, 2013.

[6] David Gay and Robion Kirby. Trisecting 4–manifolds. *Geom. Topol.*, 20(6):3097–3132, 2016.

[7] Hugh M. Hilden. Generators for two groups related to the braid group. *Pacific J. Math*, 59(2):475–486, 1975.

[8] Seiichi Kamada. *Braid and Knot Theory in Dimension Four*, volume 95 of *Mathematical Surverys and Monographs*. American Mathematical Society, 2002.

[9] Bernhard Keller. Introduction to a-infinity algebras and modules. *Homology Homotopy Appl.*, 3(1):1–35, 2001.

[10] Mikhail Khovanov. A functor-valued invariant of tangles. *Algebr. Geom. Topol.*, 2(2):665–741, 2002.

[11] Mikhail Khovanov. Patterns in knot cohomology I. *Experiment. Math.*, 12(3):365–374, 2003.

[12] K. Lefèvre-Hasegawa. Sur les A-infini catégories. *ArXiv Mathematics e-prints*, October 2003.
[13] Y. Liu. Heegaard Floer homology of surgeries on two-bridge links. *ArXiv e-prints*, February 2014.
[14] Ciprian Manolescu. An introduction to knot Floer homology. *arXiv preprint arxiv:1401.7107*, 2016.
[15] Ciprian Manolescu and Peter Ozsváth. Heegaard Floer homology and integer surgeries on links. *arxiv preprint arxiv:1011.1317v4 [math.GT]*, 2017.
[16] Jeffrey Meier and Alexander Zupan. Bridge trisections of knotted surfaces in s^4. *Trans. Amer. Math. Soc.*, (369):7343–7386, 2017.
[17] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004.
[18] Peter Ozsváth and Zoltán Szabó. Holomorphic triangles and invariants for smooth four-manifolds. *Adv. Math.*, 202(2):236–400, 2006.
[19] J. Rasmussen. Khovanov homology and the slice genus. *Inventiones Mathematicae*, 182:419–447, September 2010.
[20] Adam Saltz. Strong Khovanov-Floer theories and functoriality. *arXiv preprint arxiv:1712.08272*, 2017.
[21] Sucharit Sarkar, Cotton Seed, and Zoltan Szabó. A perturbation of the geometric spectral sequence in Khovanov homology. *arXiv preprint arxiv:1410.2877 [math.GT]*, 2016.
[22] Cotton Seed. Computations of Szabó's geometric spectral sequence in Khovanov homology. *arxiv preprint arxiv:1110.0735*, 2011.
[23] Paul Seidel. *Fukaya Categories and Picard-Lefschetz Theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society, 2008.
[24] Zoltán Szabó. A geometric spectral sequence in Khovanov homology. *Journal of Topology*, 8(4):1017 – 1044, 2015.
[25] Kokoro Tanaka. Khovanov-Jacobsson numbers and invariants of surface-knots derived from bar-natan's theory. *Proceedings of the American Mathematical Society*, 134(12):3685–3689, 2006.