A TANGLE THEORY FOR SZABÓ'S LINK HOMOLOGY

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1. Introduction

wow!

It is interesting to compare our functoriality result to Khovanov's. Let β be the identity braid. $\mathrm{CKh}(\beta)$ is equal to the rank one module over H^n, so its automorphism ring is isomorphic to the center of H^n. This leads to the crucial fact that there is only one degree zero automorphism of $\mathrm{CKh}(\beta)$. The A_∞-automorphisms of $\mathrm{CSz}(\beta)$ are classified by the *Hochschild cohomology* $HH^*(\mathcal{H}^n)$. (The center of H^n is the zeroth degree Hochschild cohomology.) Work of Rozansky [16] suggests that $HH^(\mathcal{H}^n)$ is related to the Szabó homology of torus links on n strands. In other words, the automorphisms of $\mathrm{CSz}(\beta)$ are quite complicated from an algebraic perspective. However, the automorphisms induced by tangle cobordisms are highly constrained.

There are many link homology theories which admit spectral sequences from Khovanov homology. What distinguishes Szabó's theory is that it is defined combinatorially in terms of terms of configurations. But most of our arguments do not reference configurations at all. Indeed, they apply to *Khovanov-Floer theories*: link homology theories which admit spectral sequences from Khovanov homology. These were studied by the first author, Matt Hedden, and Andrew Lobb in [3]. To avoid spectral sequence arguments and focus on chain homotopy types of maps, the first author defined *strong* Khovanov-Floer theories in [17]. The upshot is that the arguments of this paper apply to any strong Khovanov-Floer theory, including instanton knot homology and the Heegaard Floer homology of branched double covers. (The only exception is the proof that $\mathcal{H}^n$ is unital. We expect that this holds for any strong Khovanov-Floer theory.)

1.1. Connections to symplectic topology.

In light of Seed and Szabó's conjectures, it is interesting to compare our constructions with some others in symplectic topology. In unpublished work of Lekili and Perutz and upcoming work of Kotelskiy, it is shown that a reduced version of $\mathcal{H}^2$ agrees with the Fukaya category of the once-punctured torus.[1] This correspondence does not hold for $\mathcal{H}^n$, $n > 2$. John, can you fill in why? number of objects don't agree, right?

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[1]To make this precise, think of $\mathcal{H}^2$ as an A_∞-category, see Remark 3.22.

Bordered Floer homology can be used to compute the Ozsváth-Szabó spectral sequence, [1], [14], [10]. This construction assigns to each braid a bimodule over an A_∞-algebra. The particular algebra depends on the number of strands in the braid, but in the original construction of bordered Floer homology this algebra is a invariant of a (bordered) surface. Thinking in terms of branched double covers, we should compare the algebra for a genus g surface with $\mathcal{H}^{2g}$. But $\text{rank}(\mathcal{H}^2) = 12$ and the torus algebra has rank 8. Reconciling these two constructions would be a nice way to prove Seed and Szabó's conjectures.

1.2. Signs and conventions.

All of our chain complexes have coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Szabó's theory can be defined over $\mathbb{Z}$, and we expect that the tangle theory should extend to $\mathbb{Z}$ with essentially the same proofs. Our proof of functoriality is only valid on $\mathbb{F}$.

2. Background

All the tangles in this paper have even numbers of endpoints.

Definition 2.1. An (m, n)*-tangle* is a proper, smooth embedding t of a compact one-manifold with boundary into $\mathbb{R}^2 \times I$ so that

(1) $\partial t = \{1, \ldots, 2m\} \times \{0\} \times \{0\} \cup \{1, \ldots, 2n\} \times \{0\} \times \{1\}$.
(2) Near the endpoints, the arcs are perpendicular to $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$.

Let t be an (m, n)-tangle and t' be a (n, p)-tangle. Then tt' is the (m, p)-tangle given by gluing t and t' together and re-scaling. Let t be an (m, n)-tangle. Write $\bar{t}$ for the (n, m)-tangle given by reflecting t over the plane $\mathbb{R}^2 \times \{\frac{1}{2}\}$.

2.1. Szabó's link homology theory.

In [19], Szabó introduced a new link homology theory built from Khovanov homology. Here we only recall the basic notions behind it and set some notation following [18].

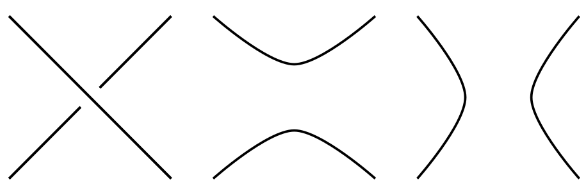


Figure 1. A crossing, its 0-resolution, and its 1-resolution.

Let $L \subset S^3$ be an oriented link and let $\mathcal{D}$ be a diagram for L. One *resolves* a crossing of $\mathcal{D}$ by replacing it with one of the two pictures shown in Figure 1. A *complete resolution* is a choice of resolution for each crossing. The two replacements are called 0 and 1, so if $\mathcal{D}$ has c crossings with some order then its resolutions are indexed by $\{0,1\}^c$, the *cube of resolutions*. For $I \in \{0,1\}^c$, write $\mathcal{D}(I)$ for the diagram given by resolving $\mathcal{D}$ according to I. Let

$$\mathrm{CSz}(\mathcal{D}(I)) = \mathrm{CKh}(\mathcal{D}(I)) \otimes \mathbb{F}[W]$$

where W is a formal variable. The *Szabó chain group of $\mathcal{D}$ is*

$$\mathrm{CSz}(\mathcal{D}) = \bigoplus_{I \in \{0,1\}^c} \mathrm{CSz}(\mathcal{D}(I)).$$

Of course this is identical to $\mathrm{CKh}(\mathcal{D}) \otimes \mathbb{F}[W]$. Recall that the Khovanov chain group has a canonical set of generators given by labeling the circles of a resolution with pluses and minuses. Call these elements *canonical generators*. $\mathrm{CSz}(\mathcal{D})$ has the same basis.The two theories differ in their differentials. Write d_1 for the Khovanov differential. The Szabó differential ∂ can be written

$$\partial = \sum_{i=1}^{\infty} W^{i-1} d_i.$$

Loosely, d_i is a map along i-dimensional diagonals of the cube of resolutions. Each such diagonal spans a face of the cube. The face can be described by the diagram at its initial vertex along with i arcs, the traces of the surgeries which transform the initial diagram into the terminal one. A trace with an orientation is called a *decoration*, and a resolution with some oriented traces is called a *configuration*. One can fix orientations on the decorations by fixing them once and for all on $\mathcal{D}(I_0)$. All of the decorations are planar: only their endpoints intersect the resolved link diagrams. See FIGURE.

Now let I and J be two resolutions of $\mathcal{D}$ with $I < J$. There is a configuration $\mathcal{C}(I,J)$ whose diagram is $\mathcal{D}(I)$. Surgery along the decorations of $\mathcal{C}(I,J)$ turn $\mathcal{D}(I)$ into $\mathcal{D}(J)$. Szabó defines a map

$$\mathcal{F}_{\mathcal{C}(I,J)} \colon \ \mathrm{CSz}(\mathcal{D}(I)) \to \mathrm{CSz}(\mathcal{D}(J)).$$

$\mathcal{F}_{\mathcal{C}(I,J)}$ is defined combinatorially by studying the topology of $\mathcal{C}(I,J)$. The exact form of $\mathcal{F}$ is not important for this paper except for the following rules. Call a circle in $\mathcal{C}(I,J)$ *active* if it meets a decoration. Otherwise, call that circle *passive*. The *active part of* $\mathcal{C}(I,J)$ is the union of all the active circles and decorations. One can identify the passive circles of $\mathcal{D}(I)$ with those of $\mathcal{D}(J)$.

- **The extension rule:** $\mathcal{F}_{\mathcal{C}(I,J)}$ must act as the identity on any tensor factor belonging to a passive circle.
- **The disconnected rule:** if the active part of $\mathcal{C}(I,J)$ is disconnected, then $\mathcal{F}_{\mathcal{C}(I,J)} = 0$.

- **The cancellation rule:** if $\mathcal{C}(I,J)$ has dimension greater than one and has an active, degree one circle, then $\mathcal{F}_{\mathcal{C}(I,J)} = 0$.

$\mathrm{CSz}(\mathcal{D})$ admits two gradings. Write n_+ and n_- for the number of positive and negative crossings in $\mathcal{D}$. For a resolution I write $\|I\|$ for the sum of its entries. Let $x \in \mathrm{CSz}(\mathcal{D})$ be a canonical generator in resolution I. Write $\tilde{q}(x)$ for the sum of the pluses and minuses in x. Define

$$h(x) = \|I\| - n_-(\mathcal{D})$$
$$q(x) = \tilde{q}(x) + \|I\| + n_+(\mathcal{D}) - 2n_-(\mathcal{D}).$$

Give W (h,q)-grading $(-1,-2)$. With these conventions, ∂ has (h,q)-degree $(1,0)$. Write $\mathrm{Sz}(\mathcal{D})$ for the homology of $\mathrm{CSz}(\mathcal{D})$.

Theorem ([19])**.** *The bigraded homology group* $\mathrm{Sz}(\mathcal{D})$ *is a link invariant.*

2.2. **Khovanov's construction for tangles.** Let $\mathfrak{M}^n$ be the set of crossingless $(0,2n)$-tangles, the *crossingless matchings*. Given $a,b \in \mathfrak{M}^n$ we can form the planar unlink $a\bar{b}$. There is a partial multiplication operation $\odot$ on such pairings defined by $a\bar{b} \odot b\bar{c} = a\bar{c}$. This operation can be realized geometrically by a simple cobordism $\Xi_b \colon \bar{b}b \to \mathrm{Id}_{2n}$. This cobordism has n 1-handles attached along arcs connecting corresponding components of $\bar{b}$ and b. See FIGURE.

Consider the set

$$H^n = \bigoplus_{a,b \in \mathfrak{M}^n} \mathrm{CKh}(a\bar{b}).$$

The operation $\odot$ induces a multiplication operation on H^n. Recall from [6] that a diagrammatic cobordism W induces a map F_W between Khovanov chain groups. (For crossingless diagrams, this map is a composition of operations from the Frobenius algebra $\mathbb{F}[X]/(X^2)$.) Let $x \in \mathrm{CKh}(a\bar{b})$ and $y \in \mathrm{CKh}(b\bar{c})$. Extend Ξ_b by the identity to obtain a cobordism

$$\Xi_b^{ac} \colon a\bar{b} \coprod b\bar{c} \to a\bar{c}.$$

Define

$$x \cdot y = F_{\Xi_b^{ac}}(x \otimes y).$$

For $x \in a\bar{b}$ and $z \in d\bar{c}$, $d \neq b$, set $x \cdot z = 0$. For a canonical generator x, define $q'(x) = q(x) - n$.

Proposition ([7])**.** H^n *is a graded unital ring with grading* $-q'$*.*

The only elements with grading zero in H^n are the all-v_+-labeled elements of $\mathrm{CKh}(a\bar{a})$. The unit is the sum of these elements.

We want to imitate this construction using Szabó homology. A naïve approach is to interpret Ξ_b^{ac} as an n-dimensional configuration. This approach has two shortcomings. The first is that the resulting multiplication will have degree $2(1-n)$ rather than zero. The second is that in many cases it is not even clear how to interpret Ξ_b^{ac} as a configuration. Every decoration coming from a cube of resolutions is embedded so that only its endpoints meet the resolved link diagram. If two arcs of b are nested, then there may be no *planar* arc representing the one-handle attachment between the inner arcs. So the construction of the previous section does not apply.

A better approach is to use higher-dimensional configurations as homotopies between different applications of Khovanov's multiplication, just as higher dimensional configurations in Szabó homology constitute homotopies between different compositions of Khovanov's maps. The resulting structure is an A_∞-algebra. The right framework to organize these maps and the ordering of the surgery arcs is that of hyperboxes.

3. Hyperboxes of chain complexes and A_∞ algebras

This section is the algebraic heart of the paper. Before diving in, here's the idea: let x, y, and z be elements of some not-necessarily-associative differential algebra A. In an A_∞-algebra (in characteristic two) there is a linear map

$$\mu_3\colon\ A^{\otimes 3} \to A$$

so that

$$(x \cdot y) \cdot z + x \cdot (y \cdot z) = d\mu_3(x \otimes y \otimes z) + \mu_3(d(x \otimes y \otimes z)). \tag{1}$$

there is a ton of redundancy with the trisections paper, all the proofs in here should go. (also the proofs are better in the trisections paper)

Multiplication in H^n is associative, and H^n has no differential. But we can interpret this equation as saying that μ_3 should represent a homotopy between the maps assigned to the cobordisms shown schematically in FIGURE. So μ_3 should be the sum of many maps which represent moving one handle past another.

The hyperbox formalism of Manolescu and Ozsváth in [13] is a nice frame for these ideas. Given a, b, c, $d \in \mathfrak{M}^n$, we construct a two-dimensional grid of diagrams showing different sequences of handle attachments. Like a cube of resolutions, the vertices are labeled by link diagrams while the edges and diagonals are labeled by configurations. The initial vertex is labeled by $a\bar{b} \coprod b\bar{c} \coprod c\bar{d}$ and the final vertex is labeled by $a\bar{d}$. Apply CSz to everything in sight to obtain a hyperbox of chain complexes. (All the differentials are zero because all the link diagrams are flat.) There is a procedure called compression which shrinks this hyperbox to a cubical complex. Let $x \in \mathrm{CSz}(a\bar{b})$, $y \in \mathrm{CSz}(b\bar{c})$, and $z \in \mathrm{CSz}(c\bar{d})$. Define $\mu_3(x \otimes y \otimes z)$ to the image of $x \otimes y \otimes z$ under the longest diagonal map of this cubical complex. Equation 1 is equivalen to the fact that the compressed hyperbox is a complex.

To summarize: to compute $\mu_3(x \otimes y \otimes z)$, one builds a hyperbox, applies CSz, compresses it, and looks at the long diagonal map. With this recipe one can construct maps

$$\mu_i\colon\ (H^n)^{\otimes i} \to H^n.$$

For these maps to comprise an A_∞-structure on H^n, they must satisfy certainly relations. So we need to construct collections of hyperboxes which are similarly related. Our goal in this section is to make all this precise.

3.1. **Hyperboxes of chain complexes.** Most of the material of this section is from [13]. Lemma 3.8, Corollary 3.10, and the tensor product material is new, and we hope it will be independently useful. We have reversed the signs of all gradings to fit with the conventions for A_∞-algebras and to reflect the fact that the Khovanov differential increases its homological and δ-gradings.

Definition 3.1. An *n-dimensional hyperbox* is a subset of $\mathbb{Z}^n$ of the form

$$[0, d_1] \times [0, d_2] \times \cdots \times [0, d_n]$$

with $d_i \geq 0$ for all i. For $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ we will write $E(\mathbf{d})$ for the corresponding hyperbox. The vector $\mathbf{d}$ is the *shape* of $E(\mathbf{d})$).

The hyperbox $E_n = E(1, \ldots, 1)$ is called a *hypercube*. A hyperbox of dimension n contains cubes of dimension less than or equal to n. Any n-dimensional $\mathbb{Z}$-vector $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ with $0 \leq \delta_i \leq d_i$ for all i represents a point in $E(\mathbf{d})$. A vector $\epsilon \in E_n$ can be thought of as a "direction vector" from δ. Let

$$\|\delta\| = \sum \delta_1 + \cdots + \delta_n.$$

Definition 3.2. An *n-dimensional hyperbox of chain complexes* of shape $\mathbf{d} \in \mathbb{Z}^n_{\geq 0}$ is a collection of graded chain complexes

$$\bigoplus_{\delta \in E(\mathbf{d})} C_\delta$$

and a collection of linear maps

$$D^\epsilon_\delta \colon C_\delta \to C_{\delta+\epsilon}$$

with $\delta \in E(\mathbf{d})$ and $\epsilon \in E_n$ satisfying the following conditions.

- If $\delta + \epsilon \notin E(\mathbf{d})$ then $D^\epsilon_\delta = 0$.
- The map $D^{\mathbf{0}}_\delta \colon C_\delta \to C_\delta$ is the differential on C_δ.
- The map D^ϵ_δ has degree $\|\epsilon\| - 1$.
- Each cube (of any dimension) is a chain complex. In other words,

$$\sum_{\epsilon + \epsilon' \leq (1, \ldots, 1)} D^{\epsilon'}_{\delta+\epsilon} \circ D^\epsilon_\delta = 0 \tag{2}$$

  for all $\delta \in E(\mathbf{d})$.

Informally, a hyperbox is a collection of cubical chain complexes stacked into a box. It is important to note that the last condition on the maps is not "the sum of all compositions is zero" – a hyperbox of complexes is not a chain complex. We collect a hypercube into $H = (C, D)$ where $C = \bigoplus C_\delta$ and $D = \bigoplus D_\delta^\epsilon$. Here are some examples.

- **A 0-dimensional hyperbox** is a chain complex.
- **A 1-dimensional hyperbox** of size (d) is a collection of chain complexes $\{C_i\}_{i=0}^d$ and chain maps $f_i \colon C_i \to C_{i+1}$. In the above notation, $C = \bigoplus_{i=0}^d C_i$ and $D = \bigoplus_{i=0}^{d-1} f_i$. That the f_i are chain maps is equivalent to equation 2.
- **A 2-dimensional hyperbox** of size (d_1, d_2) is a collection of chain complexes $\{C^{i,j}\}$ for $0 \leq i \leq d_1$ and $0 \leq j \leq d_2$ along with maps

$$f_{i,j}^{(1,0)} \colon C_{i,j} \to C_{i+1,j} \qquad \text{(horizontal maps)}$$
$$f_{i,j}^{(0,1)} \colon C_{i,j} \to C_{i,j+1} \qquad \text{(vertical maps)}$$
$$f_{i,j}^{(1,1)} \colon C_{i,j} \to C_{i+1,j+1} \qquad \text{(diagonal maps)}$$

  Equation 2 implies that the horizontal and vertical maps are chain maps. It also implies that the diagonal maps are homotopies between

$$f_H^{i,j+1} \circ f_V^{i,j}$$

  and

$$f_V^{i+1,j} \circ f_H^{i,j+1}.$$

- **A hyper<u>cube</u> of dimension** n is a cubical chain complex with diagonal maps. Such complexes underlie many spectral sequences in low-dimensional topology.

Let $H = (C, D)$ be a hyperbox of chain complexes of shape $\mathbf{d} = (d_1, \ldots, d_n)$. Our standing notation will be that $\delta \in E(\mathbf{d})$ (a "coordinate vector") and $\epsilon \in E_n$ (a "direction vector"). It will be helpful to set some more vocabulary and conventions. δ is called a *corner* if every coordinate in δ is either maximal or zero. In other words, there is some $\epsilon \in E_n$ so that

$$\delta = (d_1\epsilon_1, \ldots, d_n\epsilon_n).$$

We call δ (or the chain complex at δ) the ϵ-*corner*. We will sloppily append coordinates to a vector by writing e.g. $(\delta, 1)$ for $(\delta_1, \ldots, \delta_n, 1)$. We will write ϵ_0 and ϵ_1 for all the all-zeroes and all-ones direction vectors.

Definition 3.3. Let ${}^0H$ and ${}^1H$ be hyperboxes of chain complexes. A *map of hyperboxes* $F \colon {}^0H \to {}^1H$ is a hyperbox of size $(\mathbf{d}, 1)$ so that the $\delta_{n+1} = 0$ face of F is ${}^0H$ and the $\delta_{n+1} = 1$ face of F is ${}^1H$ with grading shifted up by 1.

A map of hyperboxes of chain complexes is determined by the edges whose $(n+1)$-st coordinate changes, i.e. a family of maps

$$F^{\epsilon}_{\delta}\colon\ {}^0C_{\delta} \to {}^1C_{\epsilon+\delta}$$

of degree $\|\epsilon\|$ satisfying

$$\sum(D^{\epsilon}_{\delta+\epsilon'} \circ F^{\epsilon'}_{\delta} + F^{\epsilon'}_{\epsilon+\delta} \circ D^{\epsilon}_{\delta}) = 0 \tag{3}$$

for all $\epsilon \in E(\mathbf{d})$, all ϵ with $(n+1)$-st coordinate 0, and all ϵ' with $(n+1)$-coordinate 1 so that $\epsilon + \epsilon' \leq \epsilon_1$. Conversely, a collection of maps from C to C' satisfying these relations defines a map of hyperboxes.

Let $F\colon {}^0H \to {}^1H$ and $G\colon {}^1H \to {}^2H$ be maps of hyperboxes. Their composition $G \circ F\colon {}^0H \to {}^2H$ is defined by

$$(G \circ F)^{\epsilon}_{\epsilon^0}\colon\ {}^0C^{\epsilon^0} \to {}^2C^{\epsilon^0+\epsilon}$$
$$(G \circ F)^{\epsilon}_{\epsilon^0} = \sum_{\epsilon' \leq \epsilon} G^{\epsilon-\epsilon'}_{\epsilon^0+\epsilon'} \circ F^{\epsilon'}_{\epsilon^0}$$

In terms of boxes: glue F and G together along their common face to obtain a hyperbox of shape $(\mathbf{d}, 2)$. To obtain a hyperbox of shape $(\mathbf{d}, 1)$, compose all possible combinations of maps in the $(n+1)$-st direction. One can give similar definitions for homotopies, homotopy equivalences, and quasi-isomorphisms in the category of hyperboxes of chain complexes. In fact, if one thinks of two maps F and G as hyperboxes, then a chain homotopy can be thought of as a map of these hyperboxes:

Definition 3.4. Let $F, F' : {}^0H \to {}^1H$. A *chain homotopy* from F to F' is a hyperbox J of size $(\mathbf{d}, 1, 1)$ so that $\delta_{n+2} = 0$ face of J is F, the $\delta_{n+2} = 1$ face of J is F', and any edge map in the direction $(0, \dots, 0, 1)$ is the identity map.

- **A map of zero-dimensional hyperboxes** is the mapping cone of a chain map.
- **A map of n-dimensional hypercubes** is an $(n+1)$-dimensional hypercube, i.e. the mapping cone of a map of cubical complexes. A chain homotopy of maps of hypercubes is equivalent to a chain homotopy of chain maps of cubical complexes.

3.1.1. *Compression.* There is a recursive recipe called *compression* for building a chain complex from a hyperbox of chain complexes. Let $H = (C, D)$ be a hyperbox of chain complexes of shape $\mathbf{d} = (d_1, \dots, d_n)$. Let $\widehat{C}$ be the hypercube whose underlying vector space is the sum of the corners of H. One can construct a differential $\widehat{D}$ on $\widehat{C}$ from H. The hypercube $\widehat{H} = (\widehat{C}, \widehat{D})$ is the *compression* of H. This recipe was first described by Manolescu and Ozsváth. Presented below is an alternative view due to Liu [11].

Let H be a one-dimensional hyperbox. Define

$$\widehat{C} = C_{\epsilon_0} \oplus C_{\epsilon_1}$$
$$\widehat{D}_0^1 = f_{n-1} \circ \cdots \circ f_0$$

and

$$\widehat{H} = (\widehat{C}, \widehat{D}).$$

Let H be an n-dimensional hyperbox with shape $(d_1, \ldots, d_n)$ and $d_n > 1$. We can think of H as d_n hyperboxes of shape $(d_1, \ldots, d_{n-1}, 1)$ attached along faces. Label these hyperboxes as $H^{n,1}$, $H^{n,2}$, and so on. Each of these boxes is a map of hyperboxes of dimension $n-1$.

Definition 3.5. Define $\widehat{H}^n$ to be

$$\widehat{H}^n = H^{n,d_n} \circ \cdots \circ H^{n,1}.$$

$\widehat{H}^n$ is the *partial compression of* H *along the* n*th axis*, or just the n-th partial compression. It has shape $(d_1, \ldots, d_{n-1}, 1)$, and

If $d_n = 1$, then $\widehat{H}^n = H$.

Definition 3.6. Let H be an n-dimensional hyperbox. Define

$$\widehat{H} = H^{n,n-1,\ldots,1}$$

In other words, $\widehat{H}$ is the result of n partial compressions starting with the nth and ending with first.

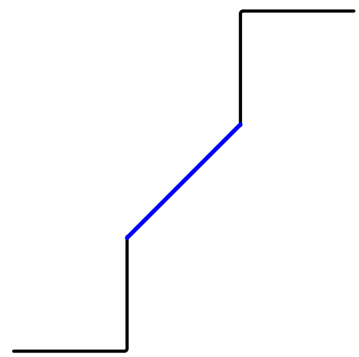
As above, if H is one-dimensional, then $\widehat{H} = H^1$ is the mapping cone of the composition of all the maps. Let H be a two-dimensional hyperbox of shape $(d_1, 2)$. H^2 is the a hyperbox of shape $(d_1, 1)$. Write $g^{\epsilon}_{(i,j)}$ for the maps in H^2. Then $g^{(1,0)}_{(i,j)}$ is the same as $f^{(1,0)}_{(i,j)}$. Next,

$$g^{(0,1)}_{(i,0)} = f^{(0,1)}_{(i,1)} \circ f^{(0,1)}_{(i,0)}.$$

To compute $g^{(1,1)}_{(i,0)}$ we think of H as two maps of one-dimensional hyperboxes. H^2 is the composition of these maps.

$$g^{(1,1)}_{(i,0)} = f^{(1,1)}_{(i,1)} \circ f^{(0,1)}_{(i,0)} + f^{(0,1)}_{(i+1,1)} \circ f^{(1,1)}_{(i,0)}$$

To compute $g^{(1,1)}_{(i,0)}$ for a hyperbox of size (d_1, d_2), we compose d_2 maps of one-dimensional hyperboxes. We can understand $g^{(1,1)}_{(i,0)}$ as follows: consider all the paths in H from $(i, 0)$ to $(i + 1, d_2)$ for which each step increases the second coordinate. Each path defines a map by composing all the maps along all its edges. $g^{(1,1)}_{(i,0)}$ is the



sum of those maps. Note that to describe one of these paths, it suffices to say which diagonal edge the path includes – the rest of the path is vertical.

Computing $\widehat{H} = H^{2,1}$ from H^2 is the same process: the vertical maps stay the same. The horizontal maps are the compositions of all the horizontal maps. The new diagonal map is the sum of maps described by paths from $(0,0)$ to $(d_1, 1)$. Each path has to move strictly rightwards. Again, each path is fully described by saying which diagonal edge it includes.

The result is fairly concrete: $\widehat{H}$ has underlying space

$$C_{(0,0)} \oplus C_{(1,0)} \oplus C_{(0,1)} \oplus C_{(1,1)}.$$

The vertical (and horizontal) maps are compositions of vertical (and horizontal) maps in H. The diagonal map is a sum of maps, one for each diagonal map in H. This diagonal completely describes a path from $(0,0)$ to (d_1, d_2) of the shape: Here the blue diagonal represents a kind of step which can only appear once. Readers familiar with hyperboxes will recognize that this description of $\widehat{H}$ agrees with Manolescu and Ozsváth's.

Proposition 3.7 ([11])**.** *Liu's definition agrees with Manolescu and Ozsváth's.*

If $G\colon H \to H'$ is a map of hyperboxes, then $\widehat{G}$ is a map $\widehat{H} \to \widehat{H'}$. Suppose that $F\colon H' \to H''$ is another map of hyperboxes. It would be nice if

$$\widehat{F \circ G} = \widehat{F} \circ \widehat{G}.$$

But this formula is false. Write FG for the hyperbox given by gluing together F and G along the appropriate faces so that $\widehat{FG}^{n+1} = F \circ G$ and

$$\widehat{FG} = \widehat{F \circ G}.$$

On the other hand, $\widehat{F} \circ \widehat{G}$ is computed by first fully compressing both F and G. In summary,

$$\widehat{F} \circ \widehat{G} = \widehat{FG}^{n,\ldots,1,n+1}$$
$$\widehat{F \circ G} = \widehat{FG}^{n+1,n,\ldots,1}.$$

Nevertheless,

Lemma 3.8. $\widehat{F \circ G} \simeq \widehat{F} \circ \widehat{G}$.

Proof. Suppose that the theorem holds if H, H', and H'' are one-dimensional. Let them instead be n-dimensional with $n > 1$ and suppose that the theorem holds for $(n-1)$-dimensional hyperboxes. We can think of each as a one-dimensional hyperbox *in the category of hyperboxes* – a hyperhyperbox. At each vertex is an $(n-1)$-dimensional hyperbox of chain complexes. So we can think of FG as a two-dimensional hyperhyperbox. The two ways to compress this hyperhyperbox yield $FG^{n+1,n}$ and $FG^{n,n+1}$. By hypothesis, these two are chain homotopy equivalent.

Now consider FG^n. We can view this as again as a two-dimensional hyperhyperbox of size $(d_{n-1}, 2)$. Now the vertices are compressed along the n-th axis and the maps are adjusted accordingly. We see that $FG^{n,n+1,n-1}$ is chain homotopy equivalent to $FG^{n,n-1,n+1}$. Continue $n-2$ more times to prove the theorem. Manolescu and Ozsváth's *algebra of songs* keeps track of what happens to individual maps in this process.

The two-dimensional claim is this: if FG is a hyperbox of size $(d, 2)$ then $FG^{2,1} \simeq FG^{1,2}$. If $d = 1$ there is nothing to show, so assume $d > 1$. The vertical and horizontal maps agree, so we only need to study the two differential diagonal maps. Call them p and q where p follows the scheme in Figure **??**. Recall that these maps are sums of maps along certain paths in the cube, one for each diagonal. Write $p_{i,j}$ and $q_{i,j}$ for the terms in $p_{i,j}$ and $q_{i,j}$ which use the diagonal from the vertex (i, j). Write $h_{i,j}$ for the map which uses the diagonals at both $(i, 0)$ and $(j, 1)$. (There is only one such path because FG has height two.) Define

$$h = \sum_{i<j} h_{i,j}$$

so that

$$\begin{aligned} h_{(i,j),(k,\ell)} \circ D^{(0,0)}_{(0,0)} + D^{(0,0)}_{(d,2)} \circ h_{(i,j),(k,\ell)} =& f^{(1,0)}_{(d-1,2)} \circ \cdots \circ f^{(1,0)}_{(j+1,2)} \\ & \circ \left(f^{(1,0)}_{(j,2)} \circ f^{(0,1)}_{(j,1)} + f^{(0,1)}_{(j+1,1)} \circ f^{(1,0)}_{(j,1)} \right) \\ & \circ f^{(1,0)}_{(j-1,1)} \circ \cdots \circ f^{(1,0)}_{(i+1,1)} \\ & \circ f^{(1,1)}_{(i,0)} \end{aligned}$$

$$
\begin{aligned}
&\circ f^{(1,0)}_{(i-1,0)} \circ \cdots \circ f^{(1,0)}_{(1,0)} \circ f^{(1,0)}_{(0,0)} \\
&+ \\
&f^{(1,0)}_{(d-1,2)} \circ \cdots \circ f^{(1,0)}_{(j+1,2)} \\
&\circ f^{(1,1)}_{(j,1)} \\
&\circ f^{(1,0)}_{(j-1,1)} \circ \cdots \circ f^{(1,0)}_{(i+1,1)} \\
&\circ \left(f^{(1,0)}_{(i,2)} \circ f^{(0,1)}_{(i,1)} + f^{(0,1)}_{(i+1,1)} \circ f^{(1,0)}_{(i,1)} \right) \\
&\circ f^{(1,0)}_{(i-1,0)} \circ \cdots \circ f^{(1,0)}_{(1,0)} \circ f^{(1,0)}_{(0,0)}
\end{aligned}
$$

This sum is easiest to understand in the visual calculus of FIGURE. From there it is straightforward to see that

$$\sum_{i<j} h_{(i,j),(k,\ell)} \circ D^{(0,0)}_{(0,0)} + D^{(0,0)}_{(d,2)} \circ h_{(i,j),(k,\ell)} = p + q$$

as in the proof of Stokes theorem from multi-variable calculus. □

BIG FIGURE

Lemma 3.9. *Suppose that $F \simeq G$ are maps of hyperboxes. Then $\widehat{F} \simeq \widehat{G}$.*

Proof. Let $J\colon F \to G$ be the homotopy. Then $\widehat{J}$ is the mapping cone of a chain map $\mathrm{Id} + j\colon \widehat{F} \to \widehat{G}$. Meanwhile $\widehat{F}$ and $\widehat{G}$ are mapping cones of chain maps f and g. The square of the differential on $\widehat{J}$ is

$$\mathrm{Id} \circ f + g \circ \mathrm{Id} + j \circ D + D \circ j.$$

□

Corollary 3.10. *Compression of hyperboxes is a functor from the homotopy category of hyperboxes to the homotopy category of chain complexes.*

3.1.2. *Two tensor products.* A one-dimensional hyperbox is a mapping cone of a factored chain map. For plain old chain maps f and g,

$$\mathrm{cone}(f \otimes g) \neq \mathrm{cone}(f) \otimes \mathrm{cone}(g).$$

Suppose that $\otimes$ is some kind of tensor product operation on hyperboxes. If $F\colon H_0 \to H_0'$ and $G\colon H_1 \to H_1'$ are maps of hyperboxes – so F and G are hyperboxes – then $F \otimes G$ is analogous to $\mathrm{cone}(f) \otimes \mathrm{cone}(G)$ rather than $\mathrm{cone}(f \otimes g)$. Therefore there are two different tensor products: $\otimes$ for hyperboxes in general and $\boxtimes$ for maps.

Let H be a hyperbox of chain complexes of dimension n and shape $\mathbf{d}$. Let H' be a hyperbox of chain complexes of dimension n' and shape $\mathbf{d}'$. Define $H \boxtimes H'$ to be the hyperbox of dimension $n+n'$ and shape $(\mathbf{d},\mathbf{d}')$ whose underlying space is

$$(C\otimes C')_{(\delta,\delta')} = C_\delta \otimes C'_{\delta'}.$$

and whose maps $D^\otimes$ are defined as follows:

$$D^{\otimes,(\epsilon,\epsilon')}_{(\delta,\delta')} = \begin{cases} D^\epsilon_\delta \otimes \mathrm{Id}_{H'_{\delta'}} & \epsilon' = (0,\dots,0) \\ \mathrm{Id}_{H_\delta} \otimes D'^{\epsilon'}_{\delta'} & \epsilon = (0,\dots,0) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.11. *$H\otimes H'$ is a hyperbox.*

Proof. Each cube of $H\otimes H'$ is a tensor product of cubical complexes. □

Lemma 3.12. *(1) $H\otimes(H'\otimes H'') = (H\otimes H')\otimes H''$.*

(2) $\widehat{H\otimes H'} = \widehat{H}\otimes\widehat{H'}$. Equivalently, the differential on $\widehat{H\otimes H'}$ has form $\widehat{D}\otimes \mathrm{Id}_{\widehat{H'}} + \mathrm{Id}_{\widehat{H}}\otimes\widehat{D'}$.

Proof. The first assertion is a straightforward verification in the spirit of Lemma 3.11.

The underlying groups of $\widehat{H\otimes H'}$ and $\widehat{H}\otimes\widehat{H'}$ are identical. The differential on $H\otimes H'$ can be written

$$D^\otimes = \mathrm{Id}\otimes D' + D\otimes \mathrm{Id}$$

where D is the differential on C and D' is the differential on C'. We claim this holds for $(H\otimes H')^{m+n,\dots,k}$ with $0\le k\le m+n$. Suppose it holds for $k=i$. To construct the next partial compression, one thinks of the boxes in the $(i-1)$st direction as maps of hyperboxes and composes them. This direction belongs to either H or H'. By hypothesis, all of the maps have the form

$$\mathrm{Id}_H\otimes g$$

or

$$f\otimes \mathrm{Id}_{H'}.$$

It follows that the differential on the fully compressed hyperbox has form $\mathrm{Id}\otimes D' + D\otimes \mathrm{Id}$. □

Any chain complex (C',d) can be thought of as a zero-dimensional hyperbox. Therefore $H\otimes C'$ is a hyperbox with the same shape as H, and

$$(C\otimes C')_\delta = C_\delta\otimes C'$$

$$D^{\otimes,\epsilon}_\delta = \begin{cases} D^0_\delta\otimes d & \epsilon=\epsilon_0 \\ D^\epsilon_\delta\otimes \mathrm{Id} & \text{otherwise} \end{cases}$$

Let $F\colon H_0 \to H_0'$ and $G\colon H_1 \to H_1'$ be maps of hyperboxes with shapes $(\mathbf{d}, 1)$ and $(\mathbf{d}', 1)$, respectively. Define $F \boxtimes G$ to be the map $H_0 \otimes H_1 \to H_0' \otimes H_1'$ defined by

$$(F \boxtimes G)^{(\epsilon,\epsilon')}_{(\epsilon_2,\epsilon_3)} = F^{\epsilon}_{\epsilon_2} \otimes G^{\epsilon'}_{\epsilon_3}.$$

Lemma 3.13. *(1) If F and G are maps of cubical complexes, then $F \boxtimes G$ is the usual tensor product of chain maps.*
(2) $F \boxtimes G$ really is a map of hyperboxes.
(3) $(F \boxtimes G) \circ (F' \boxtimes G') = (F \circ F') \boxtimes (G \circ G')$.
(4) $\mathrm{Id} \boxtimes \mathrm{Id} = \mathrm{Id}$.
(5) $\widehat{F \boxtimes G}$ is a map from $\widehat{H_0} \otimes \widehat{H_1}$ to $\widehat{H_0'} \otimes \widehat{H_1'}$.
(6) As a map of chain complexes, $\widehat{F \boxtimes G} = \widehat{F} \otimes \widehat{G}$.

Proof. The first statement is clear. The second holds because each cube which ends in the mapping direction is clearly a cube. The third and fourth also follow from thinking cube-by-cube. The fifth follows from the second point of Lemma 3.12. The sixth is essentially a consequence of the fact that

$$(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')$$

for linear maps. □

3.2. **A_∞-algebras.** See [5] for a nice introduction to A_∞-algebras and [8] for an exhaustive resource. We avoid tricky sign conventions by working over $\mathbb{F}$.

Definition. An *A_∞-algebra* (over $\mathbb{F}$) $\mathcal{A}$ is a $\mathbb{Z}$-graded vector space, also called $\mathcal{A}$, and a collection of maps

$$\mu_k\colon \mathcal{A}^{\otimes k} \to \mathcal{A}, k \geq 1$$

of degree $2 - k$ which satisfy, for each $n \geq 1$, the *A_n-relation*:

$$\sum_{i+j+k=n} \mu_{i+1+k} \left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}\right). \tag{4}$$

$\mathcal{A}$ is *unital* if there is an element $\iota \in \mathcal{A}$ so that
- $\mu_1(\iota) = 0$.
- $\mu_2(\iota, x) = \mu_2(x, \iota) = x$ for all $x \in \mathcal{A}$.
- Let $x \in \mathcal{A}^{\otimes k}$, be a simple tensor of length greater than two with a factor of ι. Then $\mu_k(x) = 0$.

For $n = 1$, equation (4) simply states that μ_1 is a differential on $\mathcal{A}$. For $n = 2$, it implies that the 'multiplication' μ_2 is a chain map on the complex $\mathcal{A} \otimes \mathcal{A}$. The $n = 3$ version implies that μ_2 is a chain homotopy between $\mu_2 \circ (\mu_2 \otimes \mathrm{Id})$ and $\mu_2 \circ (\mathrm{Id} \otimes \mu_2)$. So $\mathcal{A}$ has the structure of a differential graded algebra (dga) up to homotopy. In particular, if $\mu_i = 0$ for $i > 2$, then A can be thought of as a dga.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be A_∞-algebras. A *map of A_∞-algebras* is a collection of maps

$$f_k \colon \mathcal{A}^{\otimes k} \to \mathcal{B}$$

of degree $1-k$ which satisfy, for each $n \geq 1$,

$$\sum_{i+j+k=n} f_{i+1+j}\left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}\right) = \sum_{i_1+\cdots+i_r=n} \mu_r\left(f_{i_1} \otimes \cdots \otimes f_{i_r}\right). \tag{5}$$

The *identity map* is the map with $f_1 = \mathrm{Id}$ and $f_i = 0$ for $i > 1$.

If $f \colon \mathcal{A} \to \mathcal{B}$ and $g \colon \mathcal{B} \to \mathcal{C}$ are maps of A_∞-algebras, their composition $(g \circ f)$ is defined by

$$(g \circ f)_n = \sum_{i_1+\cdots+i_r=n} f_r(g_{i_1} \otimes \cdots \otimes g_{i_r}).$$

Definition. Let $f, g \colon \mathcal{A} \to \mathcal{B}$ be maps of A_∞-algebras. A *homotopy* between f and g is a collection of linear maps

$$h_k \colon \mathcal{A}^{\otimes k} \to \mathcal{B}$$

so that

$$f_n - g_n = \sum_{i_1+\cdots+i_r+k+j_1+\cdots+j_s=n} \mu_{r+1+s}(f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_s}) \tag{6}$$

$$+ \sum_{a+b+c=n} h_{a+1+c}(\mathrm{Id}^{\otimes a} \otimes \mu_b \otimes \mathrm{Id}^{\otimes c}) \tag{7}$$

A map which is homotopic to the identity map is called a *chain homotopy equivalence*.

3.3. **From hyperboxes to A_∞-algebras.** From a collection of hyperboxes one can construct an A_∞-algebra as described at the beginning of this section. Below we describe the construction and prove some functoriality properties.

We start with some notation for sequences and subsequences. Let $s = (s_0, \ldots, s_{k+1})$ be a sequence of natural numbers. Write $|s|$ for the length of $k+2$. Let $\{C_{ij} : i, j \in \mathbb{N}\}$ be a collection of chain complexes. Set

$$C_s = C_{s_0 s_1} \otimes C_{s_1 s_2} \otimes \cdots \otimes C_{s_k s_{k+1}}.$$

So we will often think of s as the sequence of pairs $((s_0, s_1), (s_1, s_2), \ldots, (s_k, s_{k+1}))$.

Write $s' \subset s$ if s' is a subsequence of s, i.e. there is an order-preserving injection $s' \hookrightarrow s$. There is a bijection between subsequences containing s_0 and s_{k+1} and 0-1 sequences of length $k-1$: for a 0-1 sequence ϵ, the subsequence $s(\epsilon) \subset s$ includes s_i precisely if $\epsilon_i = 0$. With these conventions,

$$s((0, \ldots, 0)) = s$$
$$s((1, \ldots, 1)) = (s_0, s_k + 1).$$

Let ϵ and ϵ' be two 0-1 sequences so that $\epsilon < \epsilon'$. Consider each maximal contiguous subsequence of 0s in ϵ which do not appear in ϵ'. For example, in

$$\epsilon = (\underline{0}, 1, \underline{0}, 0, 1, 0, 1, 0, \underline{0, 0})$$
$$\epsilon' = (1, 1, 1, 0, 1, 0, 1, 0, 1, 1)$$

the maximal subsequences are underlined. Let $c(\epsilon, \epsilon')$ be the set which contains, for each underlined sequence, all the corresponding elements of s and the surrounding ones. For example,

$$s = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$
$$s(\epsilon) = (1, \underline{2}, 4, \underline{5}, 7, 9, \underline{10, 11}, 12)$$
$$s(\epsilon') = (1, 5, 7, 9, 12)$$
$$c(\epsilon, \epsilon') = ((1, 2, 4), (4, 5, 7), (9, 10, 11, 12))$$

We call $c(\epsilon, \epsilon')$ the *contraction sequence of* ϵ *and* ϵ'. The *fixed sequence* $f(\epsilon, \epsilon')$ *of* ϵ *and* ϵ' is a sequence of pairs of elements from $s(\epsilon)$. Its elements are contiguous pairs of elements of $s(\epsilon)$ which do not appear in $c(\epsilon, \epsilon')$. So in the running example,

$$f(\epsilon, \epsilon') = ((7, 9)).$$

For a sequence $s = (s_0)$ of length one, define $C_s = (s_0 s_0)$.

3.3.1. *Systems of hyperboxes.*

Definition 3.14. Let $C = \{C_{i,j}\}$ be a collection of chain complexes indexed by $\mathbb{N} \times \mathbb{N}$. A *system of hyperboxes over* C, $\mathfrak{H}$ is an assignment of a $(k-1)$-dimensional hyperbox H_s to each sequence s with $|s| = k$ satisfying the following properties:

- The ϵ-corner of H_s is $C_{s(\epsilon)}$.
- Let F be the face of H_s between the ϵ- and ϵ'-corners. Then

$$F = \left(\bigotimes_{c' \in c(\epsilon, \epsilon')} H_{c'} \right) \otimes \left(\bigotimes_{f' \in f(\epsilon, \epsilon')} C_{f'} \right)$$

  We call $H_{c(\epsilon,\epsilon')}$ the *active part* of F and $C_{s(\epsilon'/\epsilon)^c}$ the *passive part.*

Observe that the initial corner of H_s is C_s and the terminal corner is $C_{s_0 s_k}$. The second bullet point is the *face condition.* It's a locality condition for the component maps – maps which combine one part of a tensor product (e.g. chain complexes assigned to disjoint unions of links) should not affect the other parts.

Let $x \in C^{\otimes k+1}$ be a simple tensor. We say that x is *admissible* if

$$x \in C_s$$

for some s. We say that s is the *underlying sequence* of x. It will be helpful to write $H_x = H_s$. Let $\mu_k(x) \in C_{i_0,i_{k+1}}$ be the image of x under the longest diagonal on $\widehat{H_x}$. Extend linearly to obtain a map

$$\mu_k\colon\ C^{\otimes k} \to C$$

for $k \geq 1$. If x is not admissible, then set $\mu_k(x) = 0$.

Proposition 3.15. *$(C, \{\mu_i\})$ is an A_∞-algebra.*

For a system of hyperboxes $\mathfrak{H}$ we call this A_∞-algebra $\mathcal{A}(\mathfrak{H})$.

Proof. Suppose first that x is admissible of length k. Write s for the sequence underlying x. Every diagonal from the origin of $\widehat{H_x}$ corresponds to a 0-1 sequence ϵ therefore to a contraction sequence $c(\epsilon_0, \epsilon)$. If this sequence has more than one element, then the corresponding diagonal map vanishes because of the condition of faces of a system of hyperboxes and Lemma 3.12. Therefore the differential on $\widehat{H_x}$ is a sum of the form

$$\sum \operatorname{Id} \otimes \mu \otimes \operatorname{Id}.$$

The component of $\widehat{D}^2$ which maps to $(1, \ldots, 1)$-corner of $\widehat{H_x}$ is

$$\sum_{i,j} \mu_{k-j+1} \circ \left(\operatorname{Id}^{\otimes i} \otimes \mu_j \otimes \operatorname{Id}^{\otimes(k-i-j)} \right) = 0.$$

This is precisely Equation 4. If x is not admissible, then

$$\left(\operatorname{Id}^{\otimes i} \otimes \mu_j \otimes \operatorname{Id}^{\otimes(k-i-j)} \right)(x)$$

is admissible only if μ_j is applied to a non-admissible simple tensor. So

$$\mu_{i+1+k-j} \left(\operatorname{Id}^{\otimes i} \otimes \mu_j \otimes \operatorname{Id}^{\otimes(k-i-j)} \right)(x) = 0$$

one way or another.

Lastly, it must hold that the degree of μ_i is $i-2$. μ_i is computed from the compression of an $(i-1)$-dimensional hyperbox. The compression of a hyperbox is a hyperbox, so it's diagonal has degree $i-2$. □

This construction constitutes a functor between the homotopy category of systems of hyperboxes of chain complexes and the homotopy category of A_∞-algebras. Let's construct the former category.

Definition 3.16. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be systems of hyperboxes over C and C' so that H_s and H'_s have the same shape for all s. A *map of systems of hyperboxes*, $\mathfrak{g}$, is a collection of maps of hyperboxes

$$G_s\colon\ H_s \to H'_s$$

which satisfies the following face condition. Let F be the (ϵ, ϵ')-faces of H_s. Then G_s must satisfy

$$G_s|_F = \left(\boxtimes_{s' \in c(\epsilon,\epsilon')} G'_s \right) \boxtimes \left(\boxtimes_{f' \in f(\epsilon,\epsilon')} G_{f'} \right)$$

Note that for $f' \in f(\epsilon, \epsilon')$, $G_{f'} \colon C_{f'} \to C'_{f'}$ is (the mapping cone of) an ordinary chain map. In other words, G_s acts on the passive part of H_s by chain maps.

For any $\mathfrak{H}$ there is an identity map $\mathrm{Id} \colon \mathfrak{H} \to \mathfrak{H}$ where G_s is the identity map for all s.

Definition 3.17. Let $G \colon \mathfrak{H} \to \mathfrak{H}'$ and $G' \colon \mathfrak{H}' \to \mathfrak{H}''$ be maps of systems of hyperboxes. Define

$$(G' \circ G)_s = G'_s \circ G_s.$$

Lemma 3.18. *Definition 3.17 actually defines a map of systems*

$$(G' \circ G) \colon \mathfrak{H} \to \mathfrak{H}''$$

and $(\mathrm{Id} \circ G) = G$ *and* $(G \circ \mathrm{Id}) = G$.

Definition 3.19. A *chain homotopy* between maps of systems G and G' is a collection of hyperboxes

$$J_s \colon G_s \to G'_s$$

whose length one maps are identity maps and which also satisfies the following face condition. Let F be a face of J_s in which the last component changes. Suppose that, restricting the last coordinate to zero, F is the (ϵ, ϵ') face of G_s. Then

$$J_s|_F = \left(\boxtimes_{\substack{s' \in c(\epsilon,\epsilon') \\ s' = (s''_1, \ldots, s_r)}} \bigoplus_{i=1}^{r} \left(F_{s''_1} \boxtimes \cdots \boxtimes F_{s''_{i-1}} \boxtimes J_{s''_i} \boxtimes G_{s''_{i+1}} \boxtimes \cdots \boxtimes G_{s''_r} \right) \right) \otimes \mathrm{Id}\,.$$

A map of systems $\mathfrak{g} \colon \mathfrak{H} \to \mathfrak{H}'$ induces a map of A_∞-algebras $\mathcal{A}(\mathfrak{g}) \colon \mathcal{A}(\mathfrak{H}) \to \mathcal{A}(\mathfrak{H}')$ in the following way. Let $x \in \mathcal{A}(\mathfrak{H})$ be a simple tensor of length n. There is a corresponding map of hyperboxes $G_x \colon H_x \to H'_x$. Define $g_n(x)$ to be the image of x under the longest diagonal map on $\widehat{G_x}$. Extend g_n to all of $C^{\otimes n}$ by linearity. We aim to show that $\{g_n\}_{n=1}^{\infty}$ satisfies Equation 5.

Consider the differential on $\widehat{G_x}$. Its restriction to the initial vertex of $\widehat{G_x}$ can be written as

$$\tilde{G} + \tilde{\mu}$$

where $\tilde{G}$ is the sum of all the G-wards maps and

$$\tilde{\mu} = \sum_{i,j,k} \mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}$$

The component of the square of this map which goes to the terminal vertex of $\widehat{G_x}$ is

$$\tilde{G} \circ \tilde{\mu} + \tilde{\mu} \circ \tilde{G} = \sum_i g_i \circ \tilde{\mu} + \mu_r \left(\sum_{i_1+\cdots+i_r=n} g_{i_1} \otimes \cdots \otimes g_{i_r} \right)$$

where

$$\tilde{G} = \sum_{j_1+\cdots+j_q=k} g_{j_1} \otimes \cdots \otimes g_{j_q}$$

by Definition 3.16 and Lemma 3.13, part 6.

Theorem 3.20. *Proposition 3.15 and the construction above define a functor from the homotopy category of systems of hyperboxes to the homotopy category of A_∞-algebras.*

Composition of maps A_∞-algebras is, fittingly, only associative up to homotopy. The homotopy category is an honest category.

Proof. Proposition 3.15 and the discussion above the theorem define the maps on objects and morphisms. We need to prove the following:

Suppose that $\mathfrak{f}$ and $\mathfrak{g}$ are chain homotopic maps of systems of hyperboxes. Let J_s be the homotopy between F_s and G_s. Let $x \in C_s$ be a simple tensor. Define $j_n(x)$ to be the longest diagonal map on $\widehat{J_x}$ applied to x. These maps satisfy equation 6 by Definition 3.19 and Lemma 3.13. If x is inadmissible then equation 6 holds by the same argument as Proposition 3.15. Therefore f is A_∞-chain homotopic to g.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be maps of systems so that $\mathfrak{g}' \circ \mathfrak{g}$ is defined. The map of A_∞-algebras induced by $G' \circ G$ is A_∞-chain homotopic to $g' \circ g$ by from Corollary 3.10, Definition 3.16, and Lemma 3.13, part 6.

It is straightforward to check that the identity map of systems induces the identity map of A_∞-algebras. □

Remark 3.21. For the sake of concreteness, this section only discussed systems of hyperboxes in which summands are indexed by pairs of natural numbers, but of course one could repeat the construction over any set. In the next section we will work with pairs of crossingless matchings.

Remark 3.22. For inadmissible x we defined $\mu_k(x) = 0$ for lack of better options. It might seem more natural to say that multiplication is not even defined on x. This amounts to constructing an A_∞-*category* rather than A_∞-algebra. One ought to be able to adapt all the definitions of this section to build an A_∞-category from a system of hyperboxes in which the objects are natural numbers, $\text{Hom}(i, j) = C_{i,j}$, and composition is given by the multiplication maps. Maps of systems induces A_∞-functors, and so on.

3.3.2. *A_∞-modules and based systems of hyperboxes.* A_∞-algebras act on A_∞-modules and bimodules. In this section we briefly recall the definitions of these modules and then extend the construction above.

Definition 3.23. Let $\mathcal{A}$ be an A_∞-algebra. A *right A_∞-module* over $\mathcal{A}$ is a $\mathbb{Z}$-graded vector space M and a collection of maps

$$m_k \colon M \otimes \mathcal{A}^{\otimes(k-1)} \to M$$

of degree $2-k$ which satisfy

$$\sum_{i+j+k=n} m_{i+1+k} \circ \left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}\right) + m_{1+k}(m_{i+j} \otimes \mathrm{Id}^{\otimes k}) = 0. \tag{8}$$

Left A_∞-modules are defined analogously.

Definition 3.24. Let $\mathcal{A}$ and $\mathcal{A}'$ be A_∞ algebras. An A_∞-*bimodule* over the two is a $\mathbb{Z}$-graded vector space M and a collection of maps

$$m_{i,j} \colon \mathcal{A}^{\otimes i} \otimes M \otimes \mathcal{A}'^{\otimes j} \to M$$

of degree $1-i-j$ which satisfy

$$\sum m_{\bullet,\bullet} \circ (\mathrm{Id}^{\bullet} \otimes m_{\bullet,\bullet} \otimes \mathrm{Id}^{\bullet}) + m_{0,\bullet} \circ (\mu_{\bullet,\bullet} \otimes \mathrm{Id}^{\otimes}) + m_{\bullet,0} \circ (\mathrm{Id}^{\bullet} \otimes \mu_{\bullet,\bullet}) = 0 \tag{9}$$

for sensible values of $\bullet$.

Definition 3.25. Let M and M' be A_∞-modules over $\mathcal{A}$. A *map of right A_∞-modules* is a collection of maps

$$f_k \colon M \otimes \mathcal{A}^{\otimes(k-1)} \to M$$

of degree $1-k$ which satisfy

$$\sum f_i \circ (\mathrm{Id}^{\otimes j} \otimes \mu_k \otimes \mathrm{Id}^{\otimes \ell}) + m_i \circ (f_{j+k} \otimes \mathrm{Id}^{\otimes \ell}) = 0.$$

Let N and N' be A_∞-bimodules over $\mathcal{A}$ and $\mathcal{A}'$. A *map of A_∞-bimodules* is a collection of maps

$$f_{i,j} \colon \mathcal{A}^{\otimes i} \otimes M \otimes \mathcal{A}'^{\otimes j}$$

of degree $(-i-j)$ satisfying the analogous equation.

Definition 3.26. Let M be an A_∞-bimodule over $\mathcal{A}$ and $\mathcal{B}$. Let M' be an A_∞-bimodule over $\mathcal{B}$ and $\mathcal{C}$. Define

$$M \tilde{\otimes}_{\mathcal{B}} M'$$

to be the A_∞-bimodule over $\mathcal{A}$ and $\mathcal{C}$ with underlying space

$$\bigoplus_{k=0}^{\infty} M \otimes \mathcal{B}^{\otimes k} \otimes M'.$$

Define $m_0^{\otimes,k}\colon\ M\otimes\mathcal{B}^{\otimes k}\otimes M'\to\bigoplus_{m\leq k}M\otimes\mathcal{B}^{\otimes m}\otimes M'$ by

$$m_0^{\otimes,k}=\sum_{i=0}^{k}m_{0,i}\otimes\mathrm{Id}^{\otimes k-i}+\sum_{j=0}^{k}\mathrm{Id}^{\otimes j}\otimes m_{j,0}+\sum_{\ell=0}^{k}\sum_{n=0}^{\ell}\mathrm{Id}^{\otimes(1+n)}\otimes\mu_\ell\otimes\mathrm{Id}^{\otimes(1+(k-n-\ell))}.$$

Define $m_0^{\otimes}=\sum_{i=0}^{\infty}m_0^{\otimes,i}$. Define

$$m_{i,0}^{\otimes,k}\colon\ \mathcal{A}^{\otimes i}\otimes M\otimes\mathcal{B}^{\otimes k}\otimes M'\to\bigoplus_{m\leq k}M\otimes\mathcal{B}^{\otimes m}\otimes M'$$

by

$$m_{i,0}^{\otimes,k}=\sum_{j\leq k}m_{i,j}\otimes\mathrm{Id}^{\otimes k-j+1}.$$

Define

$$m_{i,0}^{\otimes}=\sum_{k=0}^{\infty}m_{i,0}^{\otimes,k}.$$

Similarly, define

$$m_{0,i}^{\otimes,k}=\sum_{j\leq k}\mathrm{Id}^{\otimes k-j+1}\otimes m'_{j,i}$$

and

$$m_{0,i}^{\otimes}=\sum_{k=0}^{\infty}m_{0,i}^{\otimes,k}.$$

Finally, set $m_{i,j}^{\otimes}=0$ if $i,j>0$.

A *based sequence* is a finite sequence in $\mathbb{N}\cup\{0\}$ with a single entry of 0. A based subsequence of a based sequence is an order preserving injection $s'\hookrightarrow s$ whose image contains 0. contain 0. A based sequence is *one-sided* if it starts or ends with 0. Otherwise we say that it is *two-sided*.

Let s be a one-sided based sequence of length $k+1$. For concreteness, let us assume that 0 is on the left of s. There is a bijection between based subsequences of s and 0-1 sequences of length $k-1$: the subsequence $s(\epsilon)\subset s$ includes s_{i+1} precisely if $\epsilon_i=0$. This means that $s(\epsilon_0)=s$ and $s(\epsilon_1)=(0,s_{k+1})$.

Definition 3.27. Let $\mathfrak{H}$ be a system of hyperboxes over C. A *one-sided based system of hyperboxes over* $\mathfrak{H}$ consists of the following data:

- A collection of chain complexes C_{0i} for $i\in\mathbb{N}$.
- An assignment of a k-dimensional hyperbox H_s to each one-sided based sequence s with $|s|=k+1$ so that
  - The ϵ-corner of H_s is $C_{s(\epsilon)}$.

– (Same as Definition 3.14) Let F be the face of H_x between the ϵ- and ϵ'-corners. Then

$$F = \left(\boxtimes_{c' \in c(\epsilon,\epsilon')} H_{c'} \right) \otimes \left(\boxtimes_{f' \in f(\epsilon,\epsilon')} C_{f'} \right)$$

Let s be a two-sided based sequence. A based subsequence $s(\epsilon)$ defines a sequence

$$((s_{i_0}, s_{i_1}), \ldots, (s_{i_{m-1}}, s_{i_p}), (s_{i_p}, 0, s_{j_1}), (s_{j_1}, s_{j_2}), \ldots, (s_{j_{p-1}}, s_{j_p})).$$

Definition 3.28. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be systems of hyperboxes over C and C' respectively. A *two-sided based system of hyperboxes over* $\mathfrak{H}$ *and* $\mathfrak{H}'$ consists of the following data:

- A collection of chain complexes $C_{i0i'}$ for $i, i' \in \mathbb{N}$.
- An assignment of a k-dimensional hyperbox H_s to each two-sided based sequence s with $|s| = k+1$ so that
  – (Same as Definition 3.14) Let F be the face of H_x between the ϵ- and ϵ'-corners. Then

$$F = \left(\bigotimes_{c' \in c(\epsilon,\epsilon')} H_{c'} \right) \otimes \left(\bigotimes_{f' \in f(\epsilon,\epsilon')} C_{f'} \right)$$

Proposition 3.29. *A one-sided based system of hyperboxes over* $\mathfrak{H}$ *defines a left or right* A_∞*-module over* $\mathcal{A}(\mathfrak{H})$. *A two-sided based system of hyperboxes over* $\mathfrak{H}$ *defines an* A_∞*-bimodule over* $\mathcal{A}(\mathfrak{H})$ *and* $\mathcal{A}(\mathfrak{H}')$.

Proof. Same as Proposition 3.15. □

A map of based systems is defined identically to a map of systems, except that $F_{c'}$ for a non-based subsequence $c' \in c(\epsilon, \epsilon')$ is the zero map. In other words, the active part of each non-zero face consists of a single tensor factor on a based subsequence.

Definition 3.30. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be maps of one-sided based systems of hyperboxes. A *chain homotopy* between the two is a collection of hyperboxes

$$J_s \colon G_s \to G'_s$$

whose length one maps are the identity maps and whose faces satisfy the following condition. Suppose for concretess that s starts with 0. Let ϵ' be a based subsequence of ϵ. If $c(\epsilon', \epsilon)$ starts with a sequence which does not contain 0, then set J_s to the identity map. If $c(\epsilon', \epsilon)$ starts with s_0, $0 \in s_0$, then

$$J_s = J_{s_0} \boxtimes \mathrm{Id}.$$

The definition for two-sided systems is identical, *mutatis mutandis.*

Theorem 3.31. *Proposition 3.29 and the discussion above define a functor from the homotopy category of systems of based hyperboxes to the homotopy category of* A_∞*-(bi)modules.*

4. The Szabó A_∞-algebra and its modules

As a $\mathbb{F}$-vector space, define

$$\mathcal{H}^n = \bigoplus_{m,m' \in \mathfrak{M}^n} \mathrm{CSz}(m\bar{m'}).$$

Give $\mathcal{H}^n$ the *shifted δ-grading*

$$\delta' = h - \frac{q-n}{2}.$$

Of course the h-grading of any canonical generator of $\mathcal{H}^n$ is zero. The h is there to make multiplication by W into a graded map.

Theorem 4.1. *$\mathcal{H}^n$ is a unital A_∞-algebra with*

- $\mu_1 = 0$.
- *μ_2 is the Khovanov multiplication.*
- *The higher operations are defined by a system of hyperboxes and Theorem 3.15.*

Proof. Fix an enumeration of $\mathfrak{M}^n$ and write m_i for the i-th matching. Define $C_{ij} = \mathrm{CSz}(m_i\bar{m}_j)$ if $i, j \in \{1, \ldots, |\mathfrak{M}^n|\}$. Define C_{ij} to be the zero vector space otherwise.

Let η, η' be distinct arcs of a matching m. Say that $\eta < \eta'$ if the top endpoint of η' is above the top endpoint of η. Enumerate the arcs, respecting this order. This enumerates the 1-handle attachments in Ξ_m. Orient the traces of these 1-handle attachments from left to right. There is a partial order on the arcs given by $\eta \prec \eta'$ if η is nested inside η'. The order $<$ is a extension of $\prec$.

Let s be a sequence in $\mathbb{N}$ with $|s| = k+2$. We first form a k-dimensional *hyperbox of diagrams* $\mathcal{D}_s$ with shape $(n, \ldots, n)$. At the ϵ_0 corner place the diagram

$$m_0\bar{m}_1 \coprod m_1\bar{m}_2 \coprod \cdots \coprod m_k\bar{m_{k+1}}$$

At the vertex $\epsilon = (\epsilon_1, \ldots, \epsilon_{k-1})$ place the diagram given by attaching the first ϵ_i one-handles of Ξ_{m_i} to $\bar{m}_i \coprod m_i$. Sitting at this vertex, there are up to $k-1$ one-handle attachments (and therefore decorations) "up next" – the $(\epsilon_i + 1)$st handle of Ξ_{m_i} for $i \in \{0, \ldots, n-1\}$. A diagonal ϵ' from ϵ picks out $\|\epsilon'\|$ of these decorations and therefore defines a configuration $\mathcal{D}_\epsilon^{\epsilon'}$. Decorate the diagonals of $\mathcal{D}_s$ with these configurations.

Apply Szabó's construction to all the diagrams and configurations in $\mathcal{D}_s$. The cube with initial vertex δ and terminal vertex $\delta + \epsilon$-cube is the Szabó complex of the link built by changing the decorations defined by ϵ to crossings as in Figure 2. Putting aside the gradings for the moment, the result is a hyperbox of chain complexes which we call H_s.

The assignment $s \mapsto H_s$ clearly satisfies the first bullet of Definition 3.14. Let F be a face of H_s. Observe that the passive part of H_s is the Szabó complex of

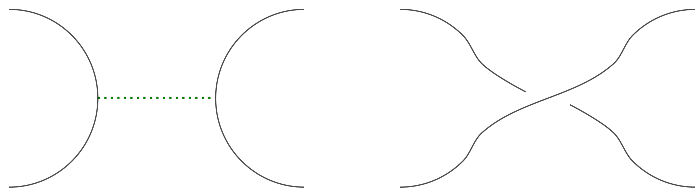


FIGURE 2. Adding a crossing where the dotted decoration is.

circles disjoint from the supports of the decorations. This implies that the edge maps restrict to the identity map on the passive part of H_s as required.

For concreteness, suppose that $c(\epsilon, \epsilon')$ has two elements, c_1 and c_2. Suppose that a diagonal ϵ'' changes coordinates in both ϵ and ϵ'. The corresponding configuration is disconnected, so by the disconnected rule for Szabó homology, the map along ϵ'' is zero. Therefore $H_{c(\epsilon,\epsilon')} = H_{c_1} \otimes H_{c_2}$, as required. More generally, if $c(\epsilon, \epsilon')$ has p elements, then $H_{c(\epsilon,\epsilon')} = H_{c_1} \otimes H_{c(\epsilon,\epsilon')\setminus\{c_1\}}$. We conclude that $s \mapsto H_s$ is a system of hyperboxes over $\mathcal{H}^n$. The internal differential on each vertex is zero because all the diagrams involved in are crossingless, so $\mu_0 = 0$. If s has length three then H_s is one-dimensional and therefore $\widehat{H_s}$ is exactly the map $F_{\Xi_{s_1}^{s_0 s_2}}$ defined by Khovanov, so μ_2 agrees with Khovanov's multiplication.

To check the grading, consider the situation in H^n. μ_2 has degree zero because $q'(x) + q'(y) = q'(\mu_2(x \otimes y))$. But most of the diagrams in the hyperbox $H_{x\otimes y}$ of are a form where q' isn't even defined. So grade C_δ by $q/2 + \|\delta\|$. Now the grading rule in Szabó homology says that C_δ is properly graded. So

$$\mu_i\colon\ (\mathcal{H}^n)^{\otimes i} \to \mathcal{H}^n[(i-1)n],$$

graded by q, is the long diagonal in a $(i-1)$-dimensional hyperbox. It therefore has degree $i-2$. It can also be understood as

$$\mu_i\colon\ (\mathcal{H}^n)^{\otimes i}[-in] \to \mathcal{H}^n[-n].$$

In other words, μ_i has degree $i-2$ under the grading q'.

Let ι be the unit in H^n. Let $x \in C^{\otimes k}$ be an admissible simple tensor of length $k > 1$ containing $\iota(m)$ for some m. Every term in this sum includes a configuration of dimension greater than one in which some circle of $\iota(m)$ is active. If this configuration is disconnected, then the corresponding map is zero. If the configuration is connected, then there is only one active circle in $\iota(m)$. Either it is degree one or it has in-degree and out-degree one. The circle is plus-labeled by definition. By the cancellation rule, such a configuration is always assigned the zero map. Therefore $\mu_k(x) = 0$. □

Remark 4.2. We reversed the signs on some gradings in Section 3 to be consistent with Khovanov's conventions. Of course we could have worked with the grading $-q'$ instead.

4.1. **Its modules.** Let t be a (p,q)-tangle diagram. Define

$$\mathrm{CSz}(t) = \bigoplus_{\substack{m \in \mathfrak{M}^p \\ m' \in \mathfrak{M}^q}} \mathrm{CSz}(mt\bar{m}')$$

by analogy with $\mathrm{CKh}(t)$. Give $\mathrm{CSz}(t)$ the shifted δ-grading

$$\delta' = h - \frac{q-n}{2}.$$

Proposition 4.3. $\mathrm{CSz}(t)$ *is a unital* A_∞*-bimodule over* $\mathcal{H}^p$*-*$\mathcal{H}^q$*.*

Proof. Assume that both p and q are greater than zero. (The other cases are easier.) Let

$$x \in \mathcal{H}^m \otimes \mathrm{CSz}(t) \otimes \mathcal{H}^n$$

be a simple tensor. In the proof of Theorem 4.1 we defined some hyperboxes of diagrams and configuration. Repeat that process; the diagrams will have all the crossings of t. Nevertheless, one can still interpret any cube of H_x as the Szabó complex of the link given by replacing the Ξ one-handles with crossings. The only difference is that the vertices have internal differentials. By exactly the same arguments, the assignment $s \mapsto H_s$ defines a two-sided based system of hyperboxes. □

Theorem 4.4. $\mathrm{CSz}(t)$ *is an invariant of tangles: if* t *and* t' *are diagrams for the same isotopy class of tangle, then* $\mathrm{CSz}(t)$ *and* $\mathrm{CSz}(t')$ *are chain homotopy equivalent as* A_∞*-bimodules by a unital chain homotopy equivalence.*

Proof. It suffices to consider the case in which t and t' are related by a single Reidemeister move. For each $m \in \mathfrak{M}^m$ and $n \in \mathfrak{M}^n$, there is a chain homotopy equivalence $\mathrm{CSz}(mt\bar{n}) \to \mathrm{CSz}(mt'\bar{n})$.

Let

$$x \in (\mathcal{H}^p)^{\otimes k} \otimes \mathrm{CSz}(mt\bar{n}) \otimes (\mathcal{H}^q)^{\otimes \ell}.$$

for some $m \in \mathfrak{M}^m, n \in \mathfrak{M}^n$. Consider the hyperbox of diagrams $\mathcal{D}_x$. In [17] the second author gave a topological construction of Szabó's chain homotopy equivalences. They can be understood as maps assigned to certain (sums of) simple cobordisms[2] with support on the interior of t. These cobordisms give rise to maps which count configurations between certain diagrams. Write $\mathcal{D}'_x$ for the hyperbox of diagrams in which each instance of t is replaced by t'. All in all, the results of [17] yield a "map"

$$\phi\colon\ \mathcal{D}_x \to \mathcal{D}'_x\,.$$

(x is not an element of $\mathrm{CSz}(t')$ but it's underlying sequence still defines $\mathcal{D}'_x$, H'_x, etc.) Apply CSz to obtain a map of hyperboxes of chain complexes

$$F_x\colon\ H_x \to H'_x.$$

[2]The cobordisms are the same used in [4] to prove that Khovanov homology is a link invariant.

By reversing the construction there are maps of hyperboxes

$$G_x\colon\ H'_x \to H_x$$

which constitute a map of based systems precisely if the F maps do. We will show that these maps form a chain homotopy equivalence, and Theorem 3.31 will complete the proof.

We verify the face condition for maps from the cobordism-theoretic description of Szabó's chain homotopy equivalences. Consider a cube of F_x whose last coordinate changes. It is the mapping cone of the sum of some cobordism maps. If that cobordism contains one-handles, then every configuration of dimension greater than one which in this cube contains the decorations corresponding to these one-handles. It follows from the disconnected rule that the (ϵ, ϵ')-face of F_x is zero if $c(\epsilon, \epsilon')$ has more than one term or if the single term is not based.

Write G for the map from the reverse Reidemeister move. Consider the map of hyperboxes

$$G_x \circ F_x\colon\ H_x \to H_x.$$

On each cube it is the mapping cone of a Reidemeister map and its inverse. Therefore there is a cubewise chain homotopy to the identity map. These compile to a map of hyperboxes

$$J_x\colon\ G_x \circ F_x \to \mathrm{Id}_x\,.$$

Just like F_x and G_x, the map J_x can be described by a collection of cobordisms with support in t. Similar arguments imply that J_x satisfies the face condition for a (based) chain homotopy. Therefore

$$G \circ F \simeq \mathrm{Id}\,.$$

□

Theorem 4.5. *Suppose that t is an (m,n)-tangle and t' is an (n,p)-tangle. Then* $\mathrm{CSz}(tt')$ *is quasi-isomorphic to* $\mathrm{CSz}(t)\tilde{\otimes}_{\mathcal{H}^n}\mathrm{CSz}(t')$*.*

Write $\mathcal{H}^n_{W=1}$ for $\mathcal{H}^n\otimes_{\mathbb{F}[W]}\mathbb{F}[W]/(W-1)$ and $\mathrm{CSz}_{W=1}(t)$ for $\mathrm{CSz}(t)\otimes_{\mathbb{F}[W]}\mathbb{F}[W]/(W-1)$. Then

$$\mathrm{CSz}_{W=1}(t)\tilde{\otimes}_{\mathcal{H}^n_{W=1}}\mathrm{CSz}_{W=1}(t') \simeq \mathrm{CSz}_{W=1}(tt').$$

Proof. First define

$$\phi_k\colon\ \mathrm{CSz}(t)\otimes(\mathcal{H}^n)^{\otimes k}\otimes\mathrm{CSz}(t') \to \mathrm{CSz}(tt').$$

as follows. Let x be a simple tensor in the domain of ϕ_k. If x is inadmissible then set $\phi_k(x) = 0$. If x is admissible, then form a hyperbox of diagrams $\mathcal{D}_x$: there are $k+1$ sets of one-handles belonging to Ξ cobordisms, ordered as above. Apply CSz to this diagram to obtain a hyperbox of chain complexes $\Phi_{k,x}$. Set $\phi_k(x)$ to be the longest diagonal map of $\widehat{\Phi_{k,x}}$ applied to x. Observe that

$$\phi_0\colon\ \mathrm{CSz}(t)\otimes\mathrm{CSz}(t')$$

is the map used above in the gluing formula for Khovanov homology (over $\mathbb{F}[W]$).

Now we extend ϕ_k to a map of bimodules. That is, we define a family of maps

$$\phi_{i|k|j}\colon\ (\mathcal{H}^m)^{\otimes i}\otimes\mathrm{CSz}(t)\otimes(\mathcal{H}^n)^{\otimes k}\otimes\mathrm{CSz}(t')\otimes(\mathcal{H}^p)^{\otimes j}\to\mathrm{CSz}(tt').$$

so that $\phi_{0|k|0}=\phi_k$. These are defined by exactly the same recipe: form a hyperbox of diagrams (of dimension $i+k+j+1$), apply Szabó's construction, compress, and look at diagonal maps. The fact that $\widehat{\Phi_{i|k|j}}$ is a complex implies a relationship between all the maps with lower indices. Write

$$x=a_1\otimes\cdots\otimes a_i\otimes y\otimes b_1\otimes\cdots\otimes b_k\otimes y'\otimes c_1\otimes\cdots\otimes c_j.$$

Then the relation is

$$\begin{aligned}\sum\phi\circ(\mathrm{Id}\otimes\mu^m\otimes\mathrm{Id})&+\phi\circ(\mathrm{Id}\otimes m\otimes\mathrm{Id})\\&+\phi\circ(\mathrm{Id}\otimes\mu^n\otimes\mathrm{Id})+\phi\circ(\mathrm{Id}\otimes m'\otimes\mathrm{Id})+\phi\circ(\mathrm{Id}\otimes\mu^p\otimes\mathrm{Id})\\&+m^{\otimes}\circ\phi\\&=0.\end{aligned}$$

where m and m' are the multiplications on $\mathrm{CSz}(t)$ and $\mathrm{CSz}(t')$, respectively; m'' is the multiplication operator on $\mathrm{CSz}(tt')$; and $\mu^\bullet$ is the multiplication on $\mathcal{H}^\bullet$. We have suppressed all the indices for digestability. This is precisely the compatability condition for an A_∞-bimodule map

$$\mathrm{CSz}(t)\tilde{\otimes}_{\mathcal{H}^n}\mathrm{CSz}(t')\to\mathrm{CSz}(tt').$$

Informally, the relationship sums up all the two-step paths to contract x to a single tensor factor. Any path must include exactly one step which combines the $\mathrm{CSz}(t)$ and $\mathrm{CSz}(t')$ factors – this is the application of ϕ.

We conclude that $\{\phi_{i|k|j}\}$ is a map of A_∞-bimodules. It is a quasi-isomorphism according to the invariance theorem of [7]. This proves the first statement.

For the second statement, appeal to the theorem of Lefevre-Hasegawa [8], following Prouté [**?**].

Theorem (Theorem 4.2.0.4, [8])**.** *Let $\mathcal{A}$ and $\mathcal{A}'$ be unital A_∞-algebras over a field. The derived category of $\mathcal{A}$-$\mathcal{A}'$-bimodules and the homotopy category of $\mathcal{A}$-$\mathcal{A}'$-bimdoules are equivalent.*

In this context, the derived category is the homotopy category in which quasi-isomorphisms are formally inverted. The theorem means that these formal inverses can be represented by actual (homotopy classes of) maps. In other words, over a field, every quasi-isomorphism has a homotopy inverse. □

4.2. **Functoriality.** The Szabó homology of links is functorial, [17]. We aim to prove the same result for the tangle invariant.

Theorem 4.6. *Let t be a (p, q)-tangle. Let Σ be a tangle cobordism from t to t'. Let W be a diagrammatic presentation of Σ. There is a map of A_∞-bimodules*

$$F_W\colon \operatorname{CSz}(t) \to \operatorname{CSz}(t')$$

so that if U is another diagrammatic presentation of Σ, then $F_W \simeq F_U$. If t is a $(0, 0)$-tangle, then F_W agrees with the maps in [17].

Let Σ' be a cobordism from t' to t'' with diagrammatic presentation W'. Then $F_{W\cup W'} = F_{W'} \circ F_W$. If W is the product cobordism, then $F_W = \mathrm{Id}$.

If W is a single zero-handle attachment, then define

$$F_{W,0,0}(x) = x \otimes v_+$$

where v_+ labels the new component. Define $F_{W,i,j} = 0$ for $i, j > 0$. If W is a two-handle attachment, then define F_W to be the dual map: write $\operatorname{CSz}(t) = \operatorname{CSz}(t') \otimes V$, write $x = x' \otimes v$, and define

$$F_{W,0,0}(x \otimes v_+) = 0$$
$$F_{W,0,0}(x \otimes v_-) = 0$$

and set $F_{W,i,j} = 0$ for $i, j > 0$. If W is a one-handle attachment, then let γ be its trace. Attach a positive crossing to t along γ as in Figure 2. Call the new diagram t^+. There is a map

$$\mathfrak{h}_\gamma\colon \operatorname{CSz}(t) \to \operatorname{CSz}(t')$$

so that[3]

$$\operatorname{CSz}(t^+) = \operatorname{cone}(\mathfrak{h}_\gamma).$$

Define

$$F_{W,0,0} = \mathfrak{h}_\gamma.$$

Define

$$F_{W,i,j}\colon (\mathcal{H}^p)^{\otimes i} \otimes \operatorname{CSz}(t) \otimes (\mathcal{H}^q)^{\otimes j} \to \operatorname{CSz}(t')$$

by the standard recipe: for a simple tensor x in the domain, draw up a hyperbox H_x. The cobordism W induces a map of hyperboxes. Take $F_{W,i,j}$ to be the longest diagonal in the compression of this map. Loosely, F_W is the module operation m except that it includes an extra decoration along γ in each configuration.

---

[3]The cone of an A_∞-morphism is exactly what you think. See Definition 2.14, [2] for a definition.

Now suppose that W is a movie, i.e a sequence $[W_1, W_2, \dots, W_k]$ where each W_i is a planar isotopy, single handle attachment, or single Reidemeister move. We have assigned maps to each of these. Define

$$F_W = F_{W_k} \circ \cdots \circ F_{W_1}.$$

Proof. If W is a single handle attachment, Reidemeister move, or planar isotopy, then the fact that F_W is a map of A_∞-bimodules follows the same argument as Theorem 4.4. For general W, F_W is a composition of these maps and is therefore itself a map. If W has no one-handles, then F_W has no higher components and the theorem follows from functoriality of Szabó homology for links.

If W has one-handles then F_W might have higher components. Let x be a simple tensor in the domain of F_W. Write H_x and H'_x for the hyperboxes underlying $F_W(x)$ and $F_U(x)$. Functoriality of Szabó homology for links implies that there is a homotopy $J_x\colon\ H_x \to H'_x$. For these homotopies to constitute a system of homotopies, they must satisfy the face condition of Definition 3.19. This follows from the disconnected rule for Szabó homology and the fact that all the chain homotopies are given by cobordisms whose support lies in t. So the diagonal corresponding to a contraction sequence with a non-based element vanishes by the disconnected rule. ☐

5. Type D structures

As noted in the introduction, $\mathrm{CSz}(t)$ is one of many tangle homology invariants which takes the form of an A_∞-bimodule over an an A_∞-algebra. The construction of $M\tilde{\otimes}_{\mathcal{A}}M'$ from Section 3.2 has underlying vector space

$$\bigoplus_{i=0}^{\infty} M \otimes \mathcal{A}^{\otimes i} \otimes M'.$$

Unfortunately, this is an infinite-dimensional vector space. This limits the utility of theorems like Theorem 4.5. Lipshitz, Ozsváth, and Thurston introduced a finite-dimensional model for A_∞-tensor products using *type D structures*.

Our goal in this section is to define type D structures for the Szabó tangle theory. Refer to [1] and [9] for details on type D structures, box tensor products[4], and so on. Throughout this section, we write $\mathcal{H}^n$ and CSz for $\mathcal{H}^n_{W=1}$ and $\mathrm{CSz}_{W=1}$, defined at the end of the previous section.

The following theorem ensures the existence of a type D structure on $\mathrm{CSz}(t)$.

Proposition 5.1 ([?], Proposition 2.3.18, Remark 2.3.28)**.** *Let $\mathcal{A}$ be an A_∞-algebra over a ring R which is unital and nilpotent. There is an equivalence of categories between the homotopy category of left A_∞-modules and the homotopy category of left type D structures over $\mathcal{A}$. The equivalence is realized as follows:*

[4]The box tensor products in this section are not related to those in Section 3, as far as we know.

- *Write A for the A_∞-algebra $\mathcal{A}$ thought of as a bimodule over itself. Let N be a left type D structure. The equivalence maps N to $A \boxtimes N$.*
- *There is a type DD structure called* $\mathrm{Bar}(A)$*, the* bar resolution *of $\mathcal{A}$, which in some sense is "$\mathcal{A}$ as a type DD structure over itself." Let M be a left A_∞-module. The equivalence maps M to* $\mathrm{Bar}(A) \boxtimes M$*.*

Let M' be a right A_∞-module over $\mathcal{A}$. Then

$$M' \tilde{\otimes} M \simeq M \boxtimes (\mathrm{Bar}(A) \boxtimes M).$$

Here is a sketch that the theorem applies to $\mathcal{H}^n$. Certainly $\mathcal{H}^n$ is unital. Think of it as an A_∞-algebra over its ring of idempotents $\mathcal{I}^n$. This ring is generated by the canonical generators which make up the unit ι. (The idempotents of H^n were identified by Khovanov in [7]. They are idempotents in $\mathcal{H}^n$ because they are the only non-negatively graded generators.) The projection $\mathcal{H}^n \to \mathcal{I}^n$ is an augmentation. The augmentation ideal is the kernel of this projection. It is negatively graded, and it follows that $\mathcal{H}^n$ is nilpotent.

Theorem 5.2. *Let t be a $(0,m)$-tangle and let t' be an $(m,0)$-tangle. There is a type D structure* $\mathrm{CSzD}(t')$ *so that* $\mathrm{CSz}_{W=1}(t) \boxtimes \mathrm{CSzD}(t') \simeq \mathrm{CSz}_{W=1}(tt')$*. The type D homotopy type of this structure is a tangle invariant.*

Proof. Combine Theorems 4.4 and **??** with Lemma **??** and Proposition 5.1. □

We admit that this is an unsatisfying state of affairs. In [12], Manion developed a type D structure on Khovanov homology. (Roberts [15] has also studied type D structures in Khovanov homology.) His recipe applies to Szabó's theory by replacing H^n with $\mathcal{H}^n$. It would be valuable to identify this type D structure with the one guaranteed to exist by Proposition 5.1.

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