INVARIANTS OF KNOTTED SURFACES FROM LINK HOMOLOGY AND BRIDGE TRISECTIONS

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1. Introduction

What can link homology say about isotopy classes? About stabilization?

questions about algebra: what else can it do, does it have interesting homology, relationship to massey products, is it cyclic

questions about q: is it 0 on non-trivial knots, connected sum, what does it count, how closely related is it to the "braidlike resolution" of the tri-plane diagram cf final computation, what is the "perturbed A_∞-algebra"

plus more on this:

Remark 1.1. From a filtered complex one can build a spectral sequence. Therefore there is a spectral sequence from $\mathrm{Kh}(\mathcal{D})$ to $\mathrm{Sz}(\mathcal{D})$. Conjecturally, this spectral sequence is isomorphic to the spectral sequence from $\mathrm{Kh}(\mathcal{D})$ to the Heegaard Floer homology of the double cover of S^3 branched along the mirror of $\mathcal{D}$ [**?**]. This conjecture was confirmed for knots up to ? crossings by Seed in []. Therefore our invariant of bridge trisections is (conjecturally) backed by Floer theory. It should provide some guide to constructing invariants of four-manifolds using Gay and Kirby's trisections and Heegaard Floer homology.

add note about branched covers of surfaces

2. Tangles

Definition. A (m, n)-*tangle* is a tangle with $2m$ left endpoints and $2n$ right endpoints.

Given an (m, n)-tangle t and an (n, p)-tangle u, one can form the (m, p)-tangle tu by concatenation. The identity braid I_{2n} is the identity elements: $tI_{2n} = t$. The *mirror* of t, denoted $\bar{t}$, is the tangle given by reflecting t over the plane $x = \frac{1}{2}$ in the unit cube. If t is an (m, n)-tangle, then $\bar{t}$ is an (n, m)-tangle. If t is a braid, then $\bar{t} = t^{-1}$.

2.1. **Braids and plat closures.** Let β be a $2n$-strand braid. Let p_n be the crossingless $(0, n)$-tangle shown in Figure 1.

Definition. The link $p_n\beta\overline{p}_n$ is called the *plat closure of* β. The $(0,2n)$-tangle $p_n\beta$, also denoted $\breve{\beta}$, is called the *half-plat closure of* β.

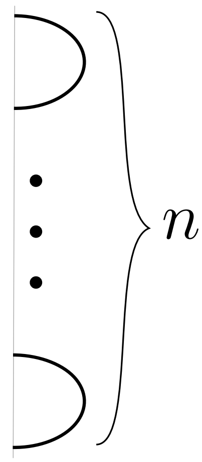

FIGURE 1. The tangle p_n.

The braid group B_{2n} acts (on the right) on the set of $(0,n)$-tangles: $t \cdot \beta = t\beta$.

Definition. The *Hilden subgroup* $H_{2n} \subset B_{2n}$ is the stabilizer of p_n.

So if $h, h' \in H_n$ and $\beta \in B_n$ then the plat closures of β and $h\beta h'$ are isotopic. Birman answered gave necessary and sufficient conditions for β and β' to have isotopic plat closures [**?**]. The situation for half-plat closures is much simpler.

Lemma 2.1. *Let* $\beta, \beta' \in B_n$. $\breve{\beta}$ *and* $\breve{\beta}'$ *are isotopic if and only if* $\beta'\beta^{-1} \in H_n$.

Proof. $p_n\beta = p_n\beta'$ if and only if $p_n\beta'\beta^{-1} = p_n$. □

Otal proved the following theorem, which can be thought of as the knot-theoretic analogue of Waldhausen's classical theorem about Heegaard splittings of connected sums of $S^1 \times S^2$s.

Theorem (Otal)**.** *Let* $\beta \in B_{2n}$. *Suppose that* $\widehat{\beta}$ *is an unlink with* k *components. Write* c_i *for the bridge number of the* i*th component of* $\widehat{\beta}$. *Then there exist* $h, h' \in H_{2n}$ *so that*

$$h\beta h' = \sigma_2\sigma_4\cdots\sigma_{2c_1}\sigma_{2c_1+4}\sigma_{2c_1+6}\cdots\sigma_{2c_1+2c_2+4}\cdots\sigma_{2c_1+\cdots+2c_k+4(k-1)}.$$

Equivalently, there is a unique bridge splitting of the k *component unlink so that the components have bridge numbers* $(c_1, \ldots, c_k)$.

Essentially, each component of $\widehat{\beta}$ appears as in Figure 2.

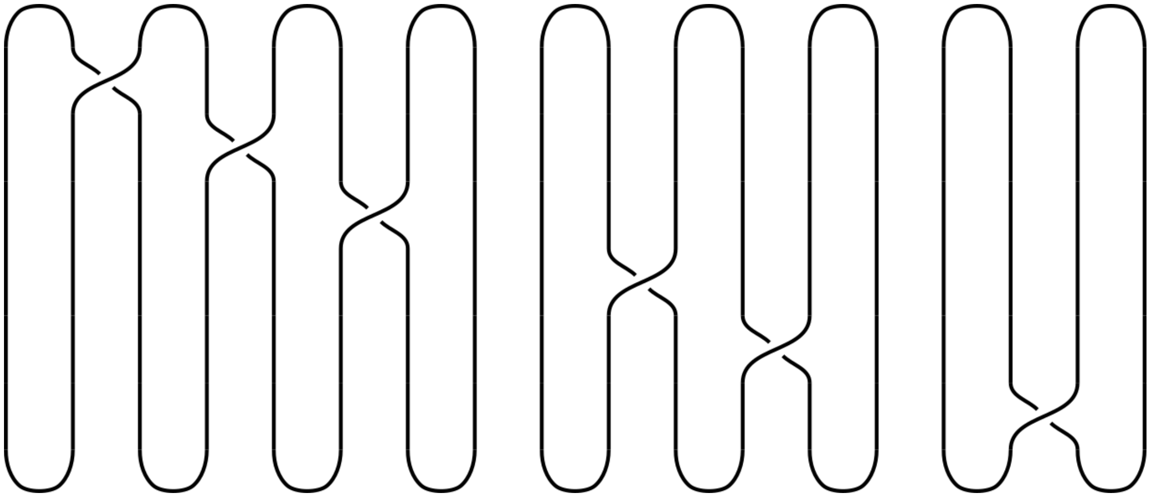

Figure 2

3. Bridge trisections and triplane diagrams

In the three-dimensional world, an *n-bridge sphere* for a knot K is a sphere S so that S cuts K into two trivial tangles and so that $S \cap K$ is a collection of n points. Here "trivial" means that all the arcs can be simultaneously isotoped to lie on S. The four-dimensional equivalent of a trivial tangle is a trivial disk system.

Definition. A *trivial c-disk system* is a pair (X, C) where X is a four-ball and $C \subset X$ is a collection of c properly embedded disks which can be simultaneously isotoped to lie on S.

A fundamental property of trivial disk systems is that (X, C) is determined up to isotopy rel boundary by the unlink $\partial X \cap C$, see [**?**]. So bisections of surfaces are not very interesting: the disk systems on each side must be identical. This, along with Gay and Kirby's trisections of four-manifolds [**?**], motivates the following definition of Meier and Zupan.

Definition. [6] A *$(b; c)$-bridge trisection* of a knotted surface $\mathcal{K} \subset S^4$ is a collection of three c-disk systems (X_1, C_1), (X_2, C_2), and (X_3, C_3), so that

- (X_1, X_2, X_3) is the standard genus 0 trisection of S^4.
- $C_1 \cup C_2 \cup C_3 = \mathcal{K}$.
- The tangle $T_{ij} = C_i \cap C_j$ is a trivial b-tangle in the three-ball $B_{ij} = X_i \cap X_j$ for all distinct i and j.

A *$(b; c_1, c_2, c_3)$-bridge trisection* is defined similarly, with (X_i, C_i) a trivial c_i-disk system.

Theorem ([6])**.** *Every knotted surface in S^4 admits a bridge trisection.*

The set $(B_{12}, T_{12}) \cup (B_{23}, T_{23}) \cup (B_{31}, T_{31})$ is called the *spine* of the trisection. Two trisections are called *isotopic* if their spines are isotopic. The sphere at the core of the unique genus 0 trisection of S^4 will be called the *bridge sphere*. The tangles T_{ij} intersect the bridge sphere in $2b$ points. Critically, $T_{ij}\bar{T}_{jk}$ is an unlink for any i,j, k. Any (cyclically ordered) triple of tangle diagrams (t_1, t_2, t_3) satisfying these two conditions is called a *triplane diagram*. By definition, every bridge trisection can be represented by a triplane diagram. Meyer and Zupan show that every triplane diagram is the triplane diagram of some bridge trisection and determine a complete set of Reidemeister-type moves for triplane diagrams.

Theorem ([6])**.** *Two triplane diagrams represent the same isotopy class of surface if and only if they are related by a sequence of the following* triplane moves*.*

Interior Reidemeister move: *a Reidemeister move on any of the three tangles performed in the complement of a neighborhood of the bridge sphere.*

Braid transposition: *the addition of an Artin generator of the braid or its inverse to the ends of all three tangles.*

Stabilization and destabilization: *Suppose that $t_1\bar{t}_2$ has a crossingless component C. Let γ be an arc so that $\partial\gamma$ lies on C, the interior of γ does not intersect $t_1\bar{t}_2$, and γ meets the bridge sphere in a single point called p. The* stabilization of $\mathbf{t}$ along γ *is the result of surgering along γ to obtain two new tangles, t_1' and t_2', then adding a small bit to t_3 at p to obtain t_3'. Destabilization is the reverse process.*

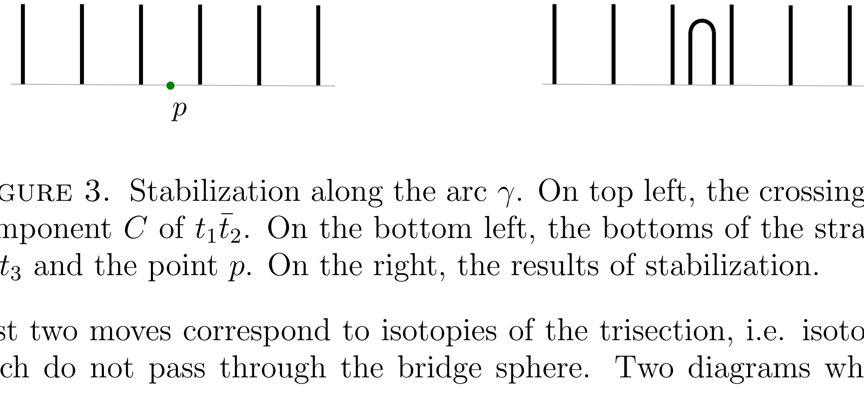

FIGURE 3. Stabilization along the arc γ. On top left, the crossingless component C of $t_1\bar{t}_2$. On the bottom left, the bottoms of the strands of t_3 and the point p. On the right, the results of stabilization.

The first two moves correspond to isotopies of the trisection, i.e. isotopies of the spine which do not pass through the bridge sphere. Two diagrams which can be related by these moves are said to lie in the same *trisection class*. Stabilization corresponds to pushing part of the surface through the spine and thus it changes the isotopy type of the trisection. In contrast to some other Reidemeister-type theorems, destabilization really is necessary: there are triplane diagrams for the same isotopy class of surface which are not isotopic after any number of stabilizations.

3.1. Orientation.

Definition. An *orientation* of a triplane diagram (t_1, t_2, t_3) is a choice of orientation on each tangle so that $t_1\overline{t_2}$, $t_2\overline{t_3}$, and $t_3\overline{t_1}$ are oriented as links.

where do we use this

Proposition. *Let* $\mathbf{t}$ *be a triplane diagram for* $\mathcal{K}$*. The set of orientations on* $\mathcal{K}$ *is in bijection with the set of orientations of* $\mathbf{t}$*.*

4. Link homology

This section introduces the link invariants which power the bridge trisection invariants following the presentation in [8].

Write V for the algebra $\mathbb{F}[X]/(X^2)$. It is standard to write v_+ for 1 and v_- for X. If $\mathcal{D}$ is a crossingless, oriented link diagram with k components, define

$$\mathrm{CKh}(\mathcal{D}) = V^{\otimes k}.$$

Concretely, $\mathrm{CKh}(\mathcal{D})$ is the vector space with a basis given by the labelings of the components of $\mathcal{D}$ by the symbols $+$ and $-$. Here $+$ corresponds to 1 and $-$ corresponds to X. We call these labelings the *canonical generators* of $\mathrm{CKh}(\mathcal{D})$.

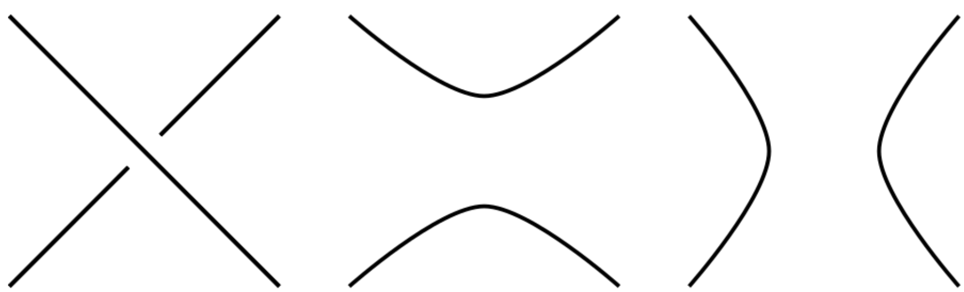

FIGURE 4. A crossing, its 0-resolution, and its 1-resolution.

Let $\mathcal{D}$ be an oriented link diagram with c crossings. There are two ways to resolve each crossing, see Figure 4. The set of resolutions of $\mathcal{D}$ is thus indexed by $\{0, 1\}^c$. For $I \in \{0, 1\}^c$ write $\mathcal{D}(I)$ for the resolution of $\mathcal{D}$ according to I. The collection of these diagrams is the *cube of resolutions*. The Khovanov chain group of $\mathcal{D}$ is defined as

$$\mathrm{CKh}(\mathcal{D}) = \bigoplus_{I\in\{0,1\}^c} \mathrm{CKh}(\mathcal{D}(I)).$$

We sometimes extend the coefficient ring from $\mathbb{F}$ to $\mathbb{F}[W]$ or $\mathbb{F}[U,W]$ where U and W are formal variables. The definition is exactly the same except that $\mathrm{CKh}(\mathcal{D}(I))$ is generated by $\mathbb{F}[U,W]$-linear combinations of the canonical generators.

There is a partial order on the cube of resolutions induced by the order on $\{0,1\}$. Write $\|I-J\|$ for the ℓ^∞ distance between I and J. If $I<J$, then $\mathcal{D}(J)$ may be obtained from $\mathcal{D}(I)$ by $\|I-J\|$ diagrammatic one-handle attachments. These one-handle attachments can be described by planar arcs in $\mathcal{D}(I)$. These arcs are called *surgery arcs*, or, if they are oriented, *decorations*. A planar diagram with k decorations is called a *k-dimensional configuration*. Orienting the surgery arcs in the all-zeroes resolution I_0 orients them in every other resolution. We will always assume that orientations of decorations on other resolutions are induced in this way.

Now make some choice of decorations for $\mathcal{D}(I_0)$. For $I<J$ and $\|I-J\|=k$ there is a k-dimensional configuration $\mathcal{C}(I,J)$ which describes how to obtain $\mathcal{D}(J)$ from $\mathcal{D}(I)$. Call the circles in $\mathcal{C}(I,J)$ which intersect decorations the *active part* of $\mathcal{C}(I,J)$. The other circles form the *passive part*. We will often conflate these circles and their labels, e.g. in the next paragraph.

To define a link homology theory, one cooks cooks up a map $F_{\mathcal{C}(I,J)}\,\mathrm{CKh}(I)\to \mathrm{CKh}(J)$ using $\mathcal{C}(I,J)$. Each of these maps acts by the identity on the passive part of $\mathrm{CKh}(I)$. In other words, write $\mathrm{CKh}(I)=\mathrm{CKh}_{\text{active}}(I)\oplus \mathrm{CKh}_{\text{passive}}(I)$; then $F_{\mathcal{C}(I,J)}$ restricts to the identity map on $\mathrm{CKh}_{\text{passive}}(I)$. This is called the *extension rule*.

Definition 4.1. Let $\mathcal{C}(I,J)$ be a k-dimensional configuration from $\mathcal{D}(I)$ to $\mathcal{D}(J)$.

- The *Khovanov configuration map*, $\mathfrak{K}_\mathcal{C}$, is defined via the Frobenius algebra structure on $\mathbb{F}[X]/(X^2)$. If $k>1$ then $\mathfrak{K}_\mathcal{C}=0$. If $k=1$, then the active part of $\mathcal{C}$ has either one or two circles. $\mathfrak{K}_\mathcal{C}$ acts by multiplication or comultiplication on the active circles.
- The *Bar-Natan configuration map*, $\mathfrak{B}_\mathcal{C}$, is defined as follows. Construct a graph whose vertices are in bijection with the active circles of $\mathcal{C}$. Put an edge between two vertices ibijection with the decorations connecting the underlying circles. Call $\mathcal{C}$ a *tree* if its graph is a union of trees. Call $\mathcal{C}$ a *dual tree* if its graph is a disjoint union of vertices connected only to themselves.

  Let $x\in \mathrm{CKh}(\mathcal{D}(I))$ be a canonical generator. $\mathfrak{B}_\mathcal{C}(x)=0$ unless $\mathcal{C}$ is a disjoint union of v_--labeled trees and v_+-labeled dual trees. On the v_--labeled trees, $\mathfrak{B}_\mathcal{C}$ is defined by

$$\mathfrak{B}_\mathcal{C}(v_-\otimes\cdots\otimes v_-)=v_-$$

and $\mathfrak{B}_{\mathcal{C}}(x) = 0$ for any other labeling. On a v_+-labeled dual tree, define

$$\mathfrak{B}_{\mathcal{C}}(1) = 1 \otimes \cdots \otimes 1$$

and $\mathfrak{B}_{\mathcal{C}}(x) = 0$ for any other labeling.

- The *Szabó configuration map*, $\mathfrak{S}_{\mathcal{C}}$, is defined in [9]. For one-dimensional configurations, $\mathfrak{S}_{\mathcal{C}} = \mathfrak{K}_{\mathcal{C}}$. It is important that $\mathfrak{S}$ satisfies the *disconnected rule:* if the union of the active part of $\mathcal{C}$ and the decorations has more than one connected component, then $\mathfrak{S}_{\mathcal{C}} = 0$.

From these maps we can construct several link homology theories. For a link diagram $\mathcal{D}$ define

$$\begin{aligned} d_{\mathrm{Kh}}, d_{\mathrm{Sz}}, d_S \colon\ & \mathrm{CKh}(\mathcal{D}) \to \mathrm{CKh}(\mathcal{D}) \\ d_{\mathrm{Kh}} &= \sum_{I<J} \mathfrak{K}_{\mathcal{C}(I,J)} \\ d_{\mathrm{Sz}} &= \sum_{I<J} W^{\|I-J\|-1} \mathfrak{S}_{\mathcal{C}(I,J)} \\ d_{\mathrm{BN}} &= \sum_{I<J} UW^{\|I-J\|-1} \mathfrak{B}_{\mathcal{C}(I,J)} \end{aligned}$$

These are the Khovanov, Szabó, and Bar-Natan differentials.[1]

$\mathrm{CKh}(\mathcal{D})$ has two gradings. Let $x \in \mathrm{CKh}(\mathcal{D}(I))$ be a canonical generator. The *homological* and *quantum gradings* of x are

$$\begin{aligned} h(x) &= \|I\| - n_- \\ q(x) &= \tilde{q}(x) + \|I\| + n_+ - 2n_-. \end{aligned}$$

Give H the (h, q)-grading $(0, -2)$. Give W the (h, q)-grading $(-1, -2)$.

Theorem ([9, 8])**.** *d_{Sz} and d_B are differentials of degree $(1, 0)$ on $\mathrm{CKh}(\mathcal{D})$. They commute (over $\mathbb{F}$) so $d_{\mathrm{Sz}} + d_{\mathrm{BN}}$ is also a differential. The graded chain homotopy types of $(\mathrm{CKh}(\mathcal{D}), d_{\mathrm{Sz}})$, $(\mathrm{CKh}(\mathcal{D}), d_{\mathrm{BN}})$, and $(\mathrm{CKh}(\mathcal{D}), d_{\mathrm{Sz}} + d_{\mathrm{BN}})$ are link invariants.*

We write $\partial = d_{\mathrm{Sz}} + d_{\mathrm{BN}}$ for the total differential. Write $\mathrm{CSz}(\mathcal{D})$ for the complex $(\mathrm{CKh}(\mathcal{D}), d_{\mathrm{Sz}})$ and $\mathrm{CS}(\mathcal{D})$ for the complex $(\mathrm{CKh}(\mathcal{D}), \partial)$. One can recover other link homology theories by setting U, W, or both to zero.

Let t and t' be $(n, 0)$-tangles so that $t\bar{t'}$ is a closed link. Given a link presented in this way, define $\mathrm{CS}(t\bar{t'})$ to be the complex above but with q-grading shifted down by n. might want to move this

[1] Bar-Natan's original construction did not consider configurations of dimension greater than 1 – $\mathfrak{B}$ is defined in [8].

4.1. **Cobordisms.** It will be important to understand how CS interacts with cobordisms of links. Let $\Sigma \subset S^3 \times I$ be a cobordism from L_0 to L_1. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be diagrams for L_0 and L_1.[2] Write Σ as a composition of *elementary cobordisms:* cylinders, handle attachments, and planar isotopies. A diagrammatic 1-handle attachment can be specified by a planar arc γ with its endpoints on $\mathcal{D}_0$. Orient this arc. Put a crossing in $\mathcal{D}_0$ along γ and call the resulting diagram $\mathcal{D}$. The 0-resolution of the new crossing yields $\mathcal{D}_0$ and the 1-resolution yields $\mathcal{D}_1$. The differential on $\mathrm{CS}(\mathcal{D})$ has a component which extends from $\mathrm{CS}(\mathcal{D}_0)$ to $\mathrm{CS}(\mathcal{D}_1)$. This is the map assigned to the 1-handle attached along γ. Call it $\mathfrak{h}_\gamma$.

0- and 2-handle attachments are much simpler. A 0-handle attachment adds a crossingless, closed component to a diagram. It is easy to show that

$$\mathrm{CS}(\mathcal{D}_0 \cup \circ) \cong \mathrm{CS}(\mathcal{D}_0) \otimes \mathrm{CS}(\circ).$$

The 0-handle attachment map is the map $\mathrm{CS}(\mathcal{D}_0) \to \mathrm{CS}(\mathcal{D})$ induced by

$$x \mapsto x \otimes v_+$$

on simple tensors. The 2-handle attachment map is the dual map induced by

$$x \otimes v_- \mapsto x.$$

In [7] we showed that CSz is *functorial*: diagrammatic descriptions of the same cobordism (up to isotopy) induce chain homotopic maps. In fact, we showed that any *conic, strong Khovanov-Floer theory* over $\mathbb{F}$ is functorial, and then we showed that CSz constitutes a conic, strong Khovanov-Floer theory.[3] CS is not a strong Khovanov-Floer theory because it does not satisfy the Künneth formula:

$$\mathrm{CS}(\mathcal{D} \coprod \mathcal{D}') \not\cong \mathrm{CS}(\mathcal{D}) \otimes \mathrm{CS}(\mathcal{D})$$

even thought they are isomorphic as $\mathbb{F}[U, W]$-modules. Also, if Σ and Σ' are cobordisms from $\mathcal{D}_0$ to $\mathcal{D}_1$ and $\mathcal{D}_0'$ to $\mathcal{D}_1'$, respectively, then

$$\mathrm{CS}(W \otimes W') \not\cong \mathrm{CS}(W) \otimes \mathrm{CS}(W').$$

Nevertheless, the proof of functoriality is valid, *mutatis mutandis*. CS satisfies every other condition to be a conic, strong Khovanov-Floer theory. And if $\mathcal{D}'$ is crossingless – i.e. if $\mathrm{CS}(\mathcal{D}')$ has a vanishing differential – then

$$\mathrm{CS}(\mathcal{D} \coprod \mathcal{D}') \cong \mathrm{CS}(\mathcal{D}) \otimes \mathrm{CS}(\mathcal{D}').$$

The Künneth formula for diagrams is only ever used in [7] in this situation. The Künneth formula for cobordisms is only used to prove the S, T, and $4Tu$ relations

[2]There is some subtlety here – see Section WHICH of [1] – but it is not relevant to this paper.

[3]We actually proved this about CSz with $W = 1$, but the proof extends to the polynomial version without trouble.

of [3]. One only needs to prove these relations for cobordisms of the form $\mathrm{Id} \otimes W'$ where Id is the product cobordism $\mathcal{D} \times I$ and W' is a cobordism of crossingless diagrams. In lieu of a Künneth formula, the cobordism maps satisfy the following:

$$F_{\mathrm{Id} \coprod W'} = G_{W'} \otimes F_{W'}$$

where G depends in some way on the topology of W'. The key observation is that $F_{W'}$ does not depend on the topology of $\mathcal{D}$! So if $F_{W'} = 0$, then $F_{\mathrm{Id} \coprod W'} = G_{W'} \otimes 0 = 0$. Therefore the S, T, and $4Tu$ relations hold for CS. Therefore the proof survives even without the Künneth formula.

Theorem 4.2. CS *is a functorial link invariant. The Reidemeister maps on* CS *are described by Bar-Natan's cobordism maps.*

Nevertheless we will see in Section 27 that CS's non-locality does make it more difficult to work with. The technical result, Proposition WHICH of [7], underlying the functoriality result has the following easy implication which will be useful in Section **??**.

improve

would be good to restate this

Proposition 4.3. *Let $\mathcal{D}$ be a link diagram. Suppose that $\mathcal{D}'$ is a subset of $\mathcal{D}$ which is planar isotopic to a (n,n)-tangle diagram. Suppose further that $\mathcal{D}'$ is isotopic, as a tangle, to the identity braid on n-strands.*

Let R and R' be two sequences of Reidemeister moves supported in $\mathcal{D}'$ which transform $\mathcal{D}'$ into Id_n. Call the resulting link diagram $\mathcal{D}''$. R and R' induce maps

$$F_R \colon \ \mathrm{CS}(\mathcal{D}) \to \mathrm{CS}(\mathcal{D}'')$$
$$F_{R'} \colon \ \mathrm{CS}(\mathcal{D}) \to \mathrm{CS}(\mathcal{D}'')$$

These maps are chain homotopic.

Proof. This holds for Bar-Natan's cobordism-theoretic link homology invariant, [3]. See for example WHICH. Theorem WHICH of [7] implies that it holds for CS. □

add

5. Hyperboxes of chain complexes and A_∞ algebras

This section is dedicated to establishing the algebraic framework for our A_∞-algebras. This framework extends the hyperbox constructions developed by Manolescu and Ozsváth in their mammoth paper on Heegaard Floer homology [5]. We show that from a collection of hyperboxes satisfying some coherence conditions one can build an A_∞-algebra. Our goal is to prove Theorem 5.18 which says that this construction is functorial (up to homotopy). Corollary **??** and Sections 5.1.2 and 5.3 are new results which we believe will be useful in future work.

good argument to move all of this to the end of the paper, there's no topology

To motivate the construction, let's first consider how one might build a differential graded algebra (dga) from a tri-plane diagram. Let

$$A(\mathbf{t}) = \bigoplus_{i,j=1}^{3} \mathrm{CKh}(t_i\bar{t}_j).$$

One can define a multiplication map

$$\mathrm{CKh}(t_i\bar{t}_j) \otimes \mathrm{CKh}(t_j\bar{t}_k) \to \mathrm{CKh}(t_i\bar{t}_j t_j\bar{t}_k) \cong \mathrm{CKh}(t_i\bar{t}_k)$$

using the cobordism shown in Figure 5.

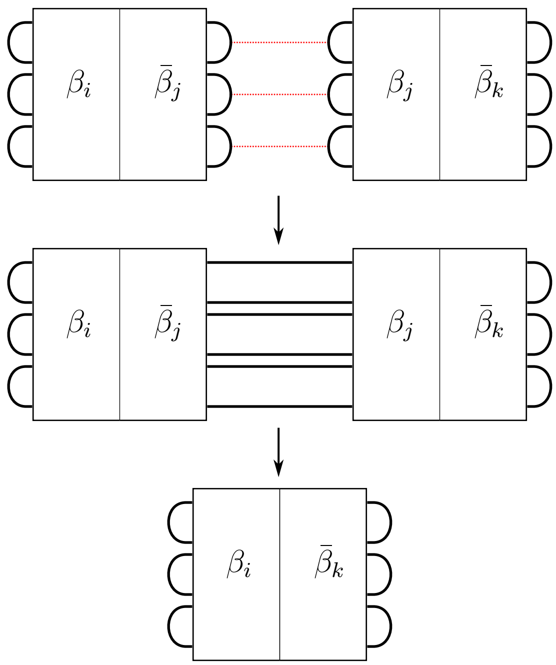

FIGURE 5. A cobordism from $t_i\bar{t}_j \coprod t_j\bar{t}_k$ to $t_i\bar{t}_k$.

Extend this to a map

$$\mu_2\colon\ A(\mathbf{t}) \otimes A(\mathbf{t}) \to A(\mathbf{t})$$

by linearity and the rule that μ_2 is zero on summands like

$$\mathrm{CKh}(t_i\bar{t}_j) \otimes \mathrm{CKh}(t_k\bar{t}_\ell), \qquad j \neq k.$$

It is not too hard (especially ignoring the gradings) to show that $A(\mathbf{t})$, equipped with the Khovanov differential and μ_2, is a dga. Now suppose one wants to define a map

$$\mu_3\colon\ A(\mathbf{t})^{\otimes 3} \to A(\mathbf{t})$$

which is a homotopy between $\mu_2 \circ (\mathrm{Id} \otimes \mu_2)$ and $\mu_2 \circ (\mu_2 \otimes \mathrm{Id})$. (Actually those two maps are equal, but forget that for a minute.) In cobordism-theoretic terms, μ_3 represents an isotopy between the two cobordisms in Figure 6. In other words, it shuffles b one-handle attachments past b other cobordisms. So it can be understood as the coallation of b^2 smaller homotopies.

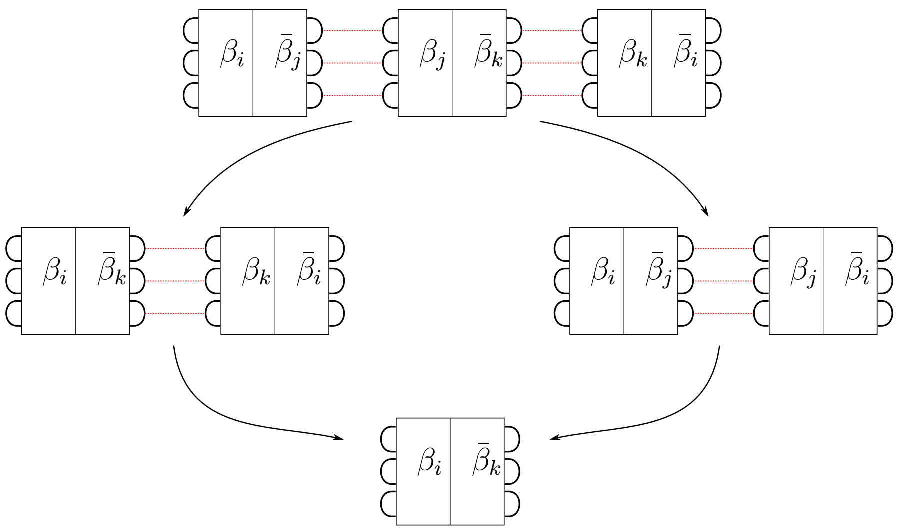

FIGURE 6. Two cobordisms from $t_i\bar{t}_j \coprod t_j\bar{t}_k \coprod t_k\bar{t}_i$ to $t_i\bar{t}_i$.

Hyperboxes are a nice way to organize these homotopies. Abstractly, the situation is this: suppose we have a trio of chain maps

$$f_0\colon\ C_0 \to C_1$$
$$f_1\colon\ C_1 \to C_2$$
$$f_2\colon\ C_2 \to C_3.$$

(For example, consider μ_2 with $b = 3$.) Consider the diagram

$$C_0 \to_{f_0} C_1 \to_{f_1} C_2 \to_{f_2} C_3.$$

This "factored mapping cone" contains at least as much information as the mapping cone of $f_2 \circ f_1 \circ f_0$. This is the definition of a one-dimensional hyperbox of chain complexes. Now suppose we have some homotopies $f_i \simeq f_i'$ for each i. Every attentive algebraic topology student knows that $f_2 \circ f_1 \circ f_0 \simeq f_2' \circ f_1' \circ f_0'$, and can draw a map between the factored mapping cones. What if these homotopies come with their own factorizations? This is the case when we move b one-handles past b others. The resulting structure is a two-dimensional hyperbox of chain complexes.

5.1. **Hyperboxes of chain complexes.** Let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$. Write $E(\mathbf{d})$ for the box with dimensions $\mathbf{d}$ and its initial corner at the origin. Write $E(n)$ for the n-dimensional box with dimensions $(1, \ldots, 1)$. We call $E(n)$ the *n-dimensional hypercube*.

Definition. An *n-dimensional hyperbox of chain complexes of shape* $\mathbf{d}$ is a collection of graded chain complexes

$$\bigoplus_{\delta \in E(\mathbf{d})} C_\delta$$

and a collection of linear maps

$$D^\epsilon_\delta \colon C^\delta \to C^{\delta+\epsilon}$$

with $\delta \in E(\mathbf{d})$ and $\epsilon \in E_n$ so that:

- If $\delta + \epsilon \notin E(\mathbf{d})$ then $D^\epsilon_\delta = 0$.
- The map $D^{\mathbf{0}}_\delta \colon C^\delta \to C^\delta$ is the differential on C^δ.
- The map D^ϵ_δ has degree $\|\epsilon\| - 1$.
- Each hypercube (of any dimension) is a chain complex. In other words,

$$\sum_{\epsilon+\epsilon' \leq (1,\ldots,1)} D^{\epsilon'}_{\delta+\epsilon} \circ D^\epsilon_\delta = 0 \tag{1}$$

  for all $\delta \in E(\mathbf{d})$.

Informally, a hyperbox is a collection of cubical chain complexes stacked into a box. It is important to note that the last condition on the maps is not "the sum of all compositions is zero" – a hyperbox of complexes is not a chain complex. We collect a hypercube into $H = (C, D)$ where $C = \bigoplus C_\delta$ and $D = \bigoplus D^\epsilon_\delta$. Here are some examples.

A 0-dimensional hyperbox is a chain complex.

A 1-dimensional hyperbox of size (d) is a collection of chain complexes $\{C_i\}_{i=0}^d$ and chain maps $f_i \colon C_i \to C_{i+1}$. In the above notation, $C = \bigoplus_{i=0}^d C_i$ and $D = \bigoplus_{i=0}^{d-1} f_i$. That the f_i are chain maps is equivalent to equation 1.

A 2-dimensional hyperbox of size (d_1, d_2) is a collection of chain complexes $\{C^{i,j}\}$ for $0 \leq i \leq d_1$ and $0 \leq j \leq d_2$ along with maps

$$f^{(1,0)}_{i,j} \colon C_{i,j} \to C_{i+1,j} \qquad \text{(horizontal maps)}$$

$$f^{(0,1)}_{i,j} \colon C_{i,j} \to C_{i,j+1} \qquad \text{(vertical maps)}$$

$$f^{(1,1)}_{i,j} \colon C_{i,j} \to C_{i+1,j+1} \qquad \text{(diagonal maps)}$$

Equation 1 implies that the horizontal and vertical maps are chain maps. It also implies that the diagonal maps are homotopies between

$$f_H^{i,j+1} \circ f_V^{i,j}$$

and

$$f_V^{i+1,j} \circ f_H^{i,j+1}.$$

A hypercube of dimension n is a cubical chain complex with diagonal maps. Such complexes underlie many spectral sequences in low-dimensional topology.

Let $H = (C, D)$ be a hyperbox of chain complexes of shape $\mathbf{d} = (d_1, \ldots, d_n)$. Our standing notation will be that $\delta \in E(\mathbf{d})$ (a "coordinate vector") and $\epsilon \in E_n$ (a "direction vector"). It will be helpful to set some more vocabulary and conventions. δ is called a *corner* if every coordinate in δ is either maximal or zero. In other words, there is some $\epsilon \in E_n$ so that

$$\delta = (d_1\epsilon_1, \ldots, d_n\epsilon_n).$$

We call δ (or the chain complex at δ) the ϵ-*corner*. We will sloppily append coordinates to a vector by writing e.g. $(\delta, 1)$ for $(\delta_1, \ldots, \delta_n, 1)$. Write ϵ_0 and ϵ_1 for the all-zeroes and all-ones direction vectors. should we tho

Definition 5.1. Let ${}^0H$ and ${}^1H$ be hyperboxes of chain complexes. A *map of hyperboxes* $F\colon {}^0H \to {}^1H$ is a hyperbox of size $(\mathbf{d}, 1)$ so that the $\delta_{n+1} = 0$ face of F is ${}^0H$ and the $\delta_{n+1} = 1$ face of F is ${}^1H$ with grading shifted up by 1.

A map of hyperboxes of chain complexes is determined by the edges whose $(n+1)$-st coordinate changes, i.e. a family of maps

$$F_\delta^\epsilon\colon {}^0C_\delta \to {}^1C_{\epsilon+\delta}$$

of degree $\|\epsilon\|$ satisfying

$$\sum (D_{\delta+\epsilon'}^\epsilon \circ F_\delta^{\epsilon'} + F_{\epsilon+\delta}^{\epsilon'} \circ D_\delta^\epsilon) = 0 \tag{2}$$

for all $\epsilon \in E(\mathbf{d})$, all ϵ with $(n+1)$-st coordinate 0, and all ϵ' with $(n+1)$-coordinate 1 so that $\epsilon + \epsilon' \leq \epsilon_1$. Conversely, a collection of maps from C to C' satisfying these relations defines a map of hyperboxes.

Let $F\colon {}^0H \to {}^1H$ and $G\colon {}^1H \to {}^2H$ be maps of hyperboxes. Their composition $G \circ F\colon {}^0H \to {}^2H$ is defined by

$$(G \circ F)_{\epsilon^0}^\epsilon\colon {}^0C^{\epsilon^0} \to {}^2C^{\epsilon^0+\epsilon} \tag{3}$$

$$(G \circ F)_{\epsilon^0}^\epsilon = \sum_{\epsilon' \leq \epsilon} G_{\epsilon^0+\epsilon'}^{\epsilon-\epsilon'} \circ F_{\epsilon^0}^{\epsilon'} \tag{4}$$

In terms of boxes: glue F and G together along their common face to obtain a hyperbox of shape $(\mathbf{d}, 2)$. To obtain a hyperbox of shape $(\mathbf{d}, 1)$, compose all possible combinations of maps in the $(n+1)$-st direction. One can give similar definitions for homotopies, homotopy equivalences, and quasi-isomorphisms in the category of hyperboxes of chain complexes. In fact, if one thinks of two maps F and G as hyperboxes, then a chain homotopy can be thought of as a map of these hyperboxes:

Definition 5.2. Let $F, F' : {}^0H \to {}^1H$. A *chain homotopy* from F to F' is a hyperbox J of size $(\mathbf{d}, 1, 1)$ so that $\delta_{n+2} = 0$ face of J is F, the $\delta_{n+2} = 1$ face of J is F', and any edge map in the direction $(0, \dots, 0, 1)$ is the identity map.

A map of zero-dimensional hyperboxes is the mapping cone of a chain map.

A map of n-dimensional hypercubes is an $(n+1)$-dimensional hypercube, i.e. the mapping cone of a map of cubical complexes. A chain homotopy of maps of hypercubes is equivalent to a chain homotopy of chain maps of cubical complexes.

5.1.1. *Compression.* There is a recursive recipe called *compression* for building a chain complex from a hyperbox of chain complexes. Let $H = (C, D)$ be a hyperbox of chain complexes of shape $\mathbf{d} = (d_1, \dots, d_n)$. Let $\widehat{C}$ be the hypercube whose underlying vector space is the sum of the corners of H. One can construct a differential $\widehat{D}$ on $\widehat{C}$ from H. The hypercube $\widehat{H} = (\widehat{C}, \widehat{D})$ is the *compression* of H. This recipe was first described by Manolescu and Ozsváth. Presented below is an alternative view due to Liu [**?**].

Let H be a one-dimensional hyperbox of shape (n). Define

$$\widehat{C} = C_0 \oplus C_n$$
$$\widehat{D}_0^1 = f_{n-1} \circ \cdots \circ f_0$$
$$\widehat{D}_0^0 = D_0^0,\ \widehat{D}_1^0 = D_n^0$$

and

$$\widehat{H} = (\widehat{C}, \widehat{D}).$$

Let H be an n-dimensional hyperbox with shape $(d_1, \dots, d_n)$ and $d_n > 1$. We can think of H as d_n hyperboxes of shape $(d_1, \dots, d_{n-1}, 1)$ attached along faces. Label these hyperboxes as $H^{n,1}$, $H^{n,2}$, and so on. Each of these boxes is a map of hyperboxes of dimension $n-1$.

Definition 5.3. Define H^n to be

$$H^n = H^{n,d_n} \circ \cdots \circ H^{n,1}.$$

H^n is the *partial compression of H along the nth axis*, or just the n-th partial compression. It has shape $(d_1, \dots, d_{n-1}, 1)$. If $d_n = 1$, then $H^n = H$.

Definition 5.4. Let H be an n-dimensional hyperbox. Define

$$\widehat{H} = H^{n,n-1,\dots,1}$$

In other words, $\widehat{H}$ is the result of n partial compressions starting with the nth and ending with first. $\widehat{H}$ is a hypercube of dimension n.

Let's describe $\widehat{H}$ more explicitly for a two-dimensional hyperbox H. First suppose that H of shape $(d, 1)$. Then $H^2 = H$, which we view as d maps of one-dimensional hyperboxes. To compute $\widehat{H} = H^{2,1}$, compose those maps. Clearly

$$\widehat{D}^{(0,1)}_{(0,0)} = D^{(0,1)}_{0,0}.$$

Next,

$$\widehat{D}^{(1,0)}_{(0,0)} = D^{(1,0)}_{d-1,0} \circ \cdots \circ D^{(1,0)}_{1,0} \circ D^{(1,0)}_{0,0}$$

To understand $\widehat{D}^{(1,1)}_{(0,0)}$ we study equation (4). The map is a sum of maps, one for each path in H^2 from the initial vertex to the terminal vertex. Such a path can only include one diagonal edge – in fact, it's totally determined by that vertex. Therefore we can describe $\widehat{D}^{(1,1)}_{(0,0)}$ by the schematic in Figure 7. The thick blue diagonal represents a kind of step which can only appear once.

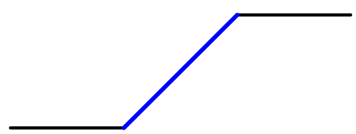

FIGURE 7

Now suppose instead that H has shape (d_1, d_2). Then H^2 is a hyperbox of shape $(d_1, 1)$. Each square in H^2 is the compression of a hyperbox of size $(1, d_2)$. Therefore its diagonal maps is given by Figure 8. Now apply the procedure above remembering that the blue diagonal stands for this shape.

The result is that $\widehat{H}$ has underlying space

$$C_{(0,0)} \oplus C_{(1,0)} \oplus C_{(0,1)} \oplus C_{(1,1)}.$$

The vertical (and horizontal) maps are compositions of vertical (and horizontal) maps in H. The diagonal map is a sum of maps, one for each diagonal map in H. This diagonal completely describes a path from $(0, 0)$ to (d_1, d_2) of the shape in Figure 9. Readers familiar with hyperboxes will recognize that this description of $\widehat{H}$ agrees with Manolescu and Ozsváth's.

Proposition 5.5 ([?])**.** *Liu's definition agrees with Manolescu and Ozsváth's.*

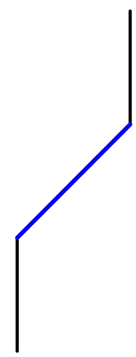

FIGURE 8

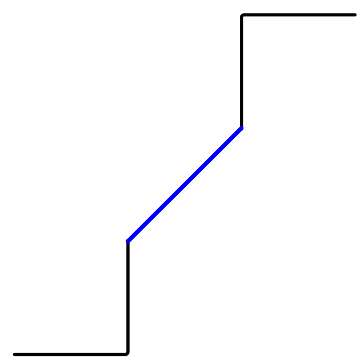

FIGURE 9

Now if $G\colon H \to H'$ is a map of hyperboxes, then $\widehat{G}$ is a map $\widehat{H} \to \widehat{H'}$. Suppose that $F\colon H' \to H''$ is another map of hyperboxes. It would be nice if

$$\widehat{F \circ G} = \widehat{F} \circ \widehat{G}.$$

But this formula is false. Write FG for the hyperbox given by gluing together F and G along the appropriate faces so that $(FG)^{n+1} = F \circ G$ and

$$\widehat{FG} = \widehat{F \circ G}.$$

On the other hand, $\widehat{F} \circ \widehat{G}$ is computed by first fully compressing both F and G. In summary,

$$\widehat{F} \circ \widehat{G} = (FG)^{n,\dots,1,n+1}$$

$$\widehat{F \circ G} = (FG)^{n+1,n,\dots,1}.$$

Nevertheless,

Lemma 5.6. *$\widehat{F \circ G} \simeq \widehat{F} \circ \widehat{G}$.*

Proof. Suppose that the theorem holds if H, H', and H'' are one-dimensional. Let them instead be n-dimensional with $n > 1$ and suppose that the theorem holds for $(n-1)$-dimensional hyperboxes. We can think of each as a one-dimensional hyperbox *in the category of hyperboxes* – a hyperhyperbox. At each vertex is an $(n-1)$-dimensional hyperbox of chain complexes. So we can think of FG as a two-dimensional hyperhyperbox. The two ways to compress this hyperhyperbox yield $FG^{n+1,n}$ and $FG^{n,n+1}$. By hypothesis, these two are chain homotopy equivalent.

Now consider FG^n. We can view this as again as a two-dimensional hyperhyperbox of size $(d_{n-1}, 2)$. Now the vertices are compressed along the n-th axis and the maps are adjusted accordingly. We see that $FG^{n,n+1,n-1}$ is chain homotopy equivalent to $FG^{n,n-1,n+1}$. Continue $n-2$ more times to prove the theorem. Manolescu and Ozsváth's *algebra of songs* keeps track of what happens to individual maps in this process.

The two-dimensional claim is this: if FG is a hyperbox of size $(d,2)$ then $FG^{2,1} \simeq FG^{1,2}$. If $d = 1$ there is nothing to show, so assume $d > 1$. The vertical and horizontal maps agree, so we only need to study the two differential diagonal maps. Call them p and q where p follows the scheme in Figure 7. Recall that these maps are sums of maps along certain paths in the cube, one for each diagonal. Write $p_{i,j}$ and $q_{i,j}$ for the terms in $p_{i,j}$ and $q_{i,j}$ which use the diagonal from the vertex (i,j). Write $h_{i,j}$ for the map which uses the diagonals at both $(i,0)$ and $(j,1)$. (There is only one such path because FG has height two.) Define

$$h = \sum_{i<j} h_{i,j}$$

so that

$$\begin{aligned}
h_{(i,j),(k,\ell)} \circ D^{(0,0)}_{(0,0)} + D^{(0,0)}_{(d,2)} \circ h_{(i,j),(k,\ell)} =& f^{(1,0)}_{(d-1,2)} \circ \cdots \circ f^{(1,0)}_{(j+1,2)} \\
& \circ \left(f^{(1,0)}_{(j,2)} \circ f^{(0,1)}_{(j,1)} + f^{(0,1)}_{(j+1,1)} \circ f^{(1,0)}_{(j,1)} \right) \\
& \circ f^{(1,0)}_{(j-1,1)} \circ \cdots \circ f^{(1,0)}_{(i+1,1)} \\
& \circ f^{(1,1)}_{(i,0)} \\
& \circ f^{(1,0)}_{(i-1,0)} \circ \cdots \circ f^{(1,0)}_{(1,0)} \circ f^{(1,0)}_{(0,0)} \\
& + \\
& f^{(1,0)}_{(d-1,2)} \circ \cdots \circ f^{(1,0)}_{(j+1,2)} \\
& \circ f^{(1,1)}_{(j,1)} \\
& \circ f^{(1,0)}_{(j-1,1)} \circ \cdots \circ f^{(1,0)}_{(i+1,1)}
\end{aligned}$$

$$\circ \left(f^{(1,0)}_{(i,2)} \circ f^{(0,1)}_{(i,1)} + f^{(0,1)}_{(i+1,1)} \circ f^{(1,0)}_{(i,1)} \right)$$
$$\circ f^{(1,0)}_{(i-1,0)} \circ \cdots \circ f^{(1,0)}_{(1,0)} \circ f^{(1,0)}_{(0,0)}$$

This sum is easiest to understand in the visual calculus of Figure 10. From there it is straightforward to see that

$$\sum_{i<j} h_{(i,j),(k,\ell)} \circ D^{(0,0)}_{(0,0)} + D^{(0,0)}_{(d,2)} \circ h_{(i,j),(k,\ell)} = p + q \tag{5}$$

as in the proof of Stokes theorem from multi-variable calculus. □

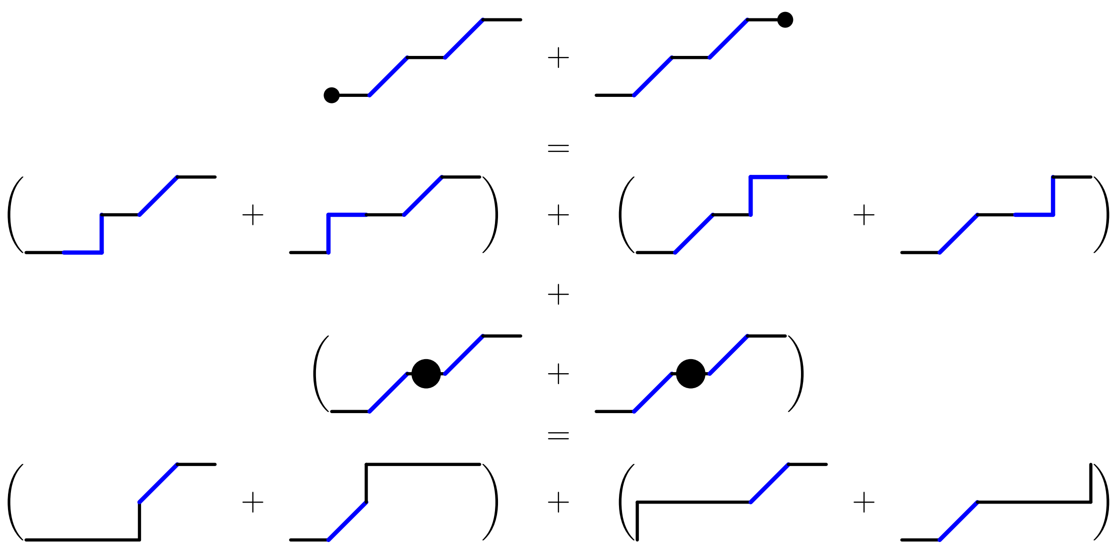

FIGURE 10. A graphical representation of equation (5). The dots represent application of the internal differential; in the musical vocabulary, they are the result of playing {}.

Lemma 5.7. *Suppose that $F \simeq G$ are maps of hyperboxes. Then $\widehat{F} \simeq \widehat{G}$.*

Proof. Let $J\colon F \to G$ be the homotopy. Then $\widehat{J}$ is the mapping cone of a chain map $\mathrm{Id} + j\colon \widehat{F} \to \widehat{G}$. Meanwhile $\widehat{F}$ and $\widehat{G}$ are mapping cones of chain maps f and g. The square of the differential on $\widehat{J}$ is

$$\mathrm{Id} \circ f + g \circ \mathrm{Id} + j \circ D + D \circ j.$$

□

Corollary 5.8. *Compression of hyperboxes is a functor from the homotopy category of hyperboxes to the homotopy category of chain complexes.*

5.1.2. *Two tensor products.* A one-dimensional hyperbox is a mapping cone of a factored chain map. For plain old chain maps f and g,

$$\mathrm{cone}(f \otimes g) \neq \mathrm{cone}(f) \otimes \mathrm{cone}(g).$$

Suppose that $\otimes$ is some kind of tensor product operation on hyperboxes. If $F\colon H_0 \to H_0'$ and $G\colon H_1 \to H_1'$ are maps of hyperboxes – so F and G are hyperboxes – then $F \otimes G$ is analogous to $\mathrm{cone}(f) \otimes \mathrm{cone}(G)$ rather than $\mathrm{cone}(f \otimes g)$. Therefore there are two different tensor products: $\otimes$ for hyperboxes in general and $\boxtimes$ for maps.

Let H be a hyperbox of chain complexes of dimension n and shape $\mathbf{d}$. Let H' be a hyperbox of chain complexes of dimension n' and shape $\mathbf{d}'$. Define $H \boxtimes H'$ to be the hyperbox of dimension $n + n'$ and shape $(\mathbf{d}, \mathbf{d}')$ whose underlying space is

$$(C \otimes C')_{(\delta, \delta')} = C_\delta \otimes C'_{\delta'}.$$

and whose maps $D^\otimes$ are defined as follows:

$$D^{\otimes,(\epsilon,\epsilon')}_{(\delta,\delta')} = \begin{cases} D^\epsilon_\delta \otimes \mathrm{Id}_{H'_{\delta'}} & \epsilon' = (0, \dots, 0) \\ \mathrm{Id}_{H_\delta} \otimes D'^{\epsilon'}_{\delta'} & \epsilon = (0, \dots, 0) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 5.9. *$H \otimes H'$ is a hyperbox.*

Proof. Each cube of $H \otimes H'$ is the ordinary tensor product of cubical complexes. ☐

Lemma 5.10. *(1) $H \otimes (H' \otimes H'') = (H \otimes H') \otimes H''$.*

(2) $\widehat{H \otimes H'} = \widehat{H} \otimes \widehat{H'}$. Equivalently, the differential on $\widehat{H \otimes H'}$ has form $\widehat{D} \otimes \mathrm{Id}_{\widehat{H'}} + \mathrm{Id}_{\widehat{H}} \otimes \widehat{D'}$.

Proof. The first assertion is a straightforward verification in the spirit of Lemma 5.9.

The underlying groups of $\widehat{H \otimes H'}$ and $\widehat{H} \otimes \widehat{H'}$ are identical. The differential on $H \otimes H'$ can be written

$$D^\otimes = \mathrm{Id} \otimes D' + D \otimes \mathrm{Id}$$

where D is the differential on C and D' is the differential on C'. We claim this holds for $(H \otimes H')^{m+n,\dots,k}$ with $0 \leq k \leq m+n$. Suppose it holds for $k = i$. To construct the next partial compression, one thinks of the boxes in the $(i-1)$st direction as maps of hyperboxes and composes them. This direction belongs to either H or H'. By hypothesis, all of the maps have the form

$$\mathrm{Id}_H \otimes g$$

or

$$f \otimes \mathrm{Id}_{H'}.$$

It follows that the differential on the fully compressed hyperbox has form $\mathrm{Id} \otimes D' + D \otimes \mathrm{Id}$. ☐

Any chain complex (C', d) can be thought of as a zero-dimensional hyperbox. Therefore $H \otimes C'$ is a hyperbox with the same shape as H and

$$(C \otimes C')_\delta = C_\delta \otimes C'$$

$$D_\delta^{\otimes,\epsilon} = \begin{cases} D_\delta^0 \otimes d & \epsilon = \epsilon_0 \\ D_\delta^\epsilon \otimes \mathrm{Id} & \text{otherwise} \end{cases}$$

Let $F\colon H_0 \to H_0'$ and $G\colon H_1 \to H_1'$ be maps of hyperboxes with shapes $(\mathbf{d}, 1)$ and $(\mathbf{d}', 1)$, respectively. Define $F \boxtimes G$ to be the map $H_0 \otimes H_1 \to H_0' \otimes H_1'$ defined by

$$(F \boxtimes G)_{(\epsilon_2,\epsilon_3)}^{(\epsilon,\epsilon')} = F_{\epsilon_2}^{\epsilon} \otimes G_{\epsilon_3}^{\epsilon'}.$$

Lemma 5.11.

(1) *If F and G are maps of cubical complexes, then $F \boxtimes G$ is the usual tensor product of chain maps.*
(2) *$F \boxtimes G$ really is a map of hyperboxes.*
(3) *$(F \boxtimes G) \circ (F' \boxtimes G') = (F \circ F') \boxtimes (G \circ G')$.*
(4) *$\mathrm{Id} \boxtimes \mathrm{Id} = \mathrm{Id}$.*
(5) *$\widehat{F \boxtimes G}$ is a map from $\widehat{H_0} \otimes \widehat{H_1}$ to $\widehat{H_0'} \otimes \widehat{H_1'}$.*
(6) *As a map of chain complexes, $\widehat{F \boxtimes G} = \widehat{F} \otimes \widehat{G}$.*

Proof. The first statement is clear. The second holds because each cube which ends in the mapping direction is clearly a cube. The third and fourth also follow from thinking cube-by-cube. The fifth follows from the second point of Lemma 5.10. The sixth is essentially a consequence of the fact that

$$(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')$$

for linear maps. □

5.2. **A_∞-algebras.** See [**?**] for a nice introduction to A_∞-algebras and [**?**] for an exhaustive resource. We avoid tricky sign conventions by working over $\mathbb{F}$.

Definition. An A_∞-*algebra* (over $\mathbb{F}$) $\mathcal{A}$ is a $\mathbb{Z}$-graded vector space, also called $\mathcal{A}$, and a collection of maps

$$\mu_k\colon \mathcal{A}^{\otimes k} \to \mathcal{A}, k \geq 1$$

of degree $2 - k$ which satisfy, for each $n \geq 1$, the A_n-*relation*:

$$\sum_{i+j+k=n} \mu_{i+1+k}\left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}\right). \tag{6}$$

$\mathcal{A}$ is *unital* if there is an element $\iota \in \mathcal{A}$ so that

- $\mu_1(\iota) = 0$.
- $\mu_2(\iota, x) = \mu_2(x, \iota) = x$ for all $x \in \tilde{\mathcal{A}}$.

- Let $x \in \tilde{\mathcal{A}}^{\otimes k}$ be a simple tensor of length greater than two with a factor of ι. Then $\mu_k(x) = 0$.

For $n = 1$, equation (6) simply states that μ_1 is a differential on $\mathcal{A}$. For $n = 2$, it implies that the multiplication μ_2 is a chain map. The $n = 3$ version implies that μ_3 is a chain homotopy between $\mu_2 \circ (\mu_2 \otimes \mathrm{Id})$ and $\mu_2 \circ (\mathrm{Id} \otimes \mu_2)$. So $\mathcal{A}$ has the structure of a dga up to homotopy. In particular, if $\mu_i = 0$ for $i > 2$, then A can be thought of as a dga.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be A_∞-algebras. A *map of A_∞-algebras* is a collection of maps

$$f_k \colon \mathcal{A}^{\otimes k} \to \mathcal{B}$$

of degree $1 - k$ which satisfy, for each $n \geq 1$,

$$\sum_{i+j+k=n} f_{i+1+j}\left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}\right) = \sum_{i_1+\cdots+i_r=n} \mu_r\left(f_{i_1} \otimes \cdots \otimes f_{i_r}\right). \tag{7}$$

The *identity map* is the map with $f_1 = \mathrm{Id}$ and $f_i = 0$ for $i > 1$.

If $f \colon \mathcal{A} \to \mathcal{B}$ and $g \colon \mathcal{B} \to \mathcal{C}$ are maps of A_∞-algebras, their composition $(g \circ f)$ is defined by

$$(g \circ f)_n = \sum_{i_1+\cdots+i_r=n} f_r(g_{i_1} \otimes \cdots \otimes g_{i_r}).$$

Definition. Let $f, g \colon \mathcal{A} \to \mathcal{B}$ be maps of A_∞-algebras. A *homotopy* between f and g is a collection of linear maps

$$h_k \colon \mathcal{A}^{\otimes k} \to \mathcal{B}$$

so that

$$f_n - g_n = \sum_{i_1+\cdots+i_r+k+j_1+\cdots+j_s=n} \mu_{r+1+s}(f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_s}) \tag{8}$$

$$+ \sum_{a+b+c=n} h_{a+1+c}(\mathrm{Id}^{\otimes a} \otimes \mu_b \otimes \mathrm{Id}^{\otimes c}) \tag{9}$$

A map which is homotopic to the identity map is called a *chain homotopy equivalence.*

5.3. **From hyperboxes to A_∞-algebras.** From a collection of hyperboxes one can construct an A_∞-algebra. Below we describe the construction and prove some functoriality properties.

First, some notation for sequences and subsequences. Let $s = (s_0, \ldots, s_{k+1})$ be a sequence of natural numbers. Write $|s|$ for the length of $k + 2$. Let $\{C_{ij} : i, j \in \mathbb{N}\}$ be a collection of chain complexes. Set

$$C_s = C_{s_0 s_1} \otimes C_{s_1 s_2} \otimes \cdots \otimes C_{s_k s_{k+1}}.$$

So we will often think of s as the sequence of pairs $((s_0, s_1), (s_1, s_2), \ldots, (s_k, s_{k+1}))$.

Write $s' \subset s$ if s' is a subsequence of s, i.e. there is an order-preserving injection $s' \hookrightarrow s$. There is a bijection between subsequences containing s_0 and s_{k+1} and 0-1 sequences of length $k-1$: for a 0-1 sequence ϵ, the subsequence $s(\epsilon) \subset s$ includes s_i precisely if $\epsilon_i = 0$. With these conventions,

$$s((0,\ldots,0)) = s$$
$$s((1,\ldots,1)) = (s_0, s_k+1).$$

Let ϵ and ϵ' be two 0-1 sequences so that $\epsilon < \epsilon'$. Consider each maximal contiguous subsequence of 0s in ϵ which do not appear in ϵ'. For example, in

$$\epsilon = (\underline{0}, 1, \underline{0}, 0, 1, 0, 1, 0, \underline{0, 0})$$
$$\epsilon' = (1, 1, 1, 0, 1, 0, 1, 0, 1, 1)$$

the maximal subsequences are underlined. Let $c(\epsilon, \epsilon')$ be the set which contains, for each underlined sequence, all the corresponding elements of s and the surrounding ones. For example,

$$s = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$
$$s(\epsilon) = (1, \underline{2}, 4, \underline{5}, 7, 9, \underline{10, 11}, 12)$$
$$s(\epsilon') = (1, 5, 7, 9, 12)$$
$$c(\epsilon, \epsilon') = ((1, 2, 4), (4, 5, 7), (9, 10, 11, 12))$$

We call $c(\epsilon, \epsilon')$ the *contraction sequence of* ϵ *and* ϵ'. The *fixed sequence* $f(\epsilon, \epsilon')$ *of* ϵ *and* ϵ' is a sequence of pairs of elements from $s(\epsilon)$. Its elements are contiguous pairs of elements of $s(\epsilon)$ which do not appear in $c(\epsilon, \epsilon')$. So in the running example,

$$f(\epsilon, \epsilon') = ((7, 9)).$$

For a sequence $s = (s_0)$ of length one, define $C_s = (s_0 s_0)$.

5.3.1. *Systems of hyperboxes.*

Definition 5.12. Let $C = \{C_{i,j}\}$ be a collection of chain complexes indexed by $\mathbb{N} \times \mathbb{N}$. A *system of hyperboxes over* C, $\mathcal{H}$ is an assignment of a $(k-1)$-dimensional hyperbox H_s to each sequence s with $|s| = k$ satisfying the following properties:

- The ϵ-corner of H_s is $C_{s(\epsilon)}$.
- Let F be the face of H_s between the ϵ- and ϵ'-corners. Then

$$F = \left(\bigotimes_{c' \in c(\epsilon, \epsilon')} H_{c'} \right) \otimes \left(\bigotimes_{f' \in f(\epsilon, \epsilon')} C_{f'} \right)$$

We call $H_{c(\epsilon, \epsilon')}$ the *active part* of F and $C_{s(\epsilon'/\epsilon)^c}$ the *passive part.*

Observe that the initial corner of H_s is C_s and the terminal corner is $C_{s_0 s_k}$. The second bullet point is the *face condition.* It's a locality condition for the component maps – maps which combine one part of a tensor product (e.g. chain complexes assigned to disjoint unions of links) should not affect the other parts.

Let $x \in C^{\otimes k+1}$ be a simple tensor. We say that x is *admissible* if

$$x \in C_s$$

for some s. We say that s is the *underlying sequence* of x. It will be helpful to write $H_x = H_s$. Let $\mu_k(x) \in C_{i_0, i_{k+1}}$ be the image of x under the longest diagonal on $\widehat{H_x}$. Extend linearly to obtain a map

$$\mu_k \colon C^{\otimes k} \to C$$

for $k \geq 1$. If x is not admissible, then set $\mu_k(x) = 0$.

Proposition 5.13. *$(C, \{\mu_i\})$ is an A_∞-algebra.*

For a system of hyperboxes $\mathcal{H}$ we call this A_∞-algebra $\tilde{\mathcal{A}}(\mathcal{H})$.

Proof. Suppose first that x is admissible of length k. Write s for the sequence underlying x. Every diagonal from the origin of $\widehat{H_x}$ corresponds to a 0-1 sequence ϵ therefore to a contraction sequence $c(\epsilon_0, \epsilon)$. If this sequence has more than one element, then the corresponding diagonal map vanishes because of the condition of faces of a system of hyperboxes and Lemma 5.10. Therefore the differential on $\widehat{H_x}$ is a sum of the form

$$\sum \mathrm{Id} \otimes \mu \otimes \mathrm{Id}\,.$$

The component of $\widehat{D}^2$ which maps to $(1, \ldots, 1)$-corner of $\widehat{H_x}$ is

$$\sum_{i,j} \mu_{k-j+1} \circ \left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes (k-i-j)}\right) = 0.$$

This is precisely Equation 6. If x is not admissible, then

$$\left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes (k-i-j)}\right)(x)$$

is admissible only if μ_j is applied to a non-admissible simple tensor. So

$$\mu_{i+1+k-j} \left(\mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes (k-i-j)}\right)(x) = 0$$

one way or another.

Lastly, it must hold that the degree of μ_i is $i-2$. μ_i is computed from the compression of an $(i-1)$-dimensional hyperbox. The compression of a hyperbox is a hyperbox, so it's diagonal has degree $i-2$. ∎

This construction constitutes a functor between the homotopy category of systems of hyperboxes of chain complexes and the homotopy category of A_∞-algebras. Let's construct the former category.

Definition 5.14. Let $\mathcal{H}$ and $\mathcal{H}'$ be systems of hyperboxes over C and C' so that H_s and H'_s have the same shape for all s. A *map of systems of hyperboxes*, $\mathfrak{g}$, is a collection of maps of hyperboxes

$$G_s\colon H_s \to H'_s$$

which satisfies the following face condition. Let F be the (ϵ, ϵ')-faces of H_s. Then G_s must satisfy

$$G_s|_F = \left(\boxtimes_{s' \in c(\epsilon,\epsilon')} G'_s \right) \boxtimes \left(\boxtimes_{f' \in f(\epsilon,\epsilon')} G_{f'} \right)$$

Note that for $f' \in f(\epsilon, \epsilon')$, $G_{f'}\colon C_{f'} \to C'_{f'}$ is (the mapping cone of) an ordinary chain map. In other words, G_s acts on the passive part of H_s by chain maps.

For any $\mathcal{H}$ there is an identity map $\mathrm{Id}\colon \mathcal{H} \to \mathcal{H}$ where G_s is the identity map for all s.

Definition 5.15. Let $G\colon \mathcal{H} \to \mathcal{H}'$ and $G'\colon \mathcal{H}' \to \mathcal{H}''$ be maps of systems of hyperboxes. Define

$$(G' \circ G)_s = G'_s \circ G_s.$$

Lemma 5.16. *Definition 5.15 actually defines a map of systems*

$$(G' \circ G)\colon \mathcal{H} \to \mathcal{H}''$$

and $(\mathrm{Id} \circ G) = G$ *and* $(G \circ \mathrm{Id}) = G$.

Definition 5.17. A *chain homotopy* between maps of systems G and G' is a collection of hyperboxes

$$J_s\colon G_s \to G'_s$$

whose length one maps are identity maps and which also satisfies the following face condition. Let F be a face of J_s in which the last component changes. Suppose that, restricting the last coordinate to zero, F is the (ϵ, ϵ') face of G_s. Then

$$J_s|_F = \left(\boxtimes_{\substack{s' \in c(\epsilon,\epsilon') \\ s' = (s''_1, \ldots, s_r)}} \bigoplus_{i=1}^{r} \left(F_{s''_1} \boxtimes \cdots \boxtimes F_{s''_{i-1}} \boxtimes J_{s''_i} \boxtimes G_{s''_{i+1}} \boxtimes \cdots \boxtimes G_{s''_r} \right) \right) \otimes \mathrm{Id}\,. \tag{10}$$

A map of systems $\mathfrak{g}\colon \mathcal{H} \to \mathcal{H}'$ induces a map of A_∞-algebras $\tilde{\mathcal{A}}(\mathfrak{g})\colon \tilde{\mathcal{A}}(\mathcal{H}) \to \tilde{\mathcal{A}}(\mathcal{H}')$ in the following way. Let $x \in \tilde{\mathcal{A}}(\mathcal{H})$ be a simple tensor of length n. There is a corresponding map of hyperboxes $G_x\colon H_x \to H'_x$. Define $g_n(x)$ to be the image of

x under the longest diagonal map on $\widehat{G_x}$. Extend g_n to all of $C^{\otimes n}$ by linearity. We aim to show that $\{g_n\}_{n=1}^{\infty}$ satisfies Equation 7.

Consider the differential on $\widehat{G_x}$. Its restriction to the initial vertex of $\widehat{G_x}$ can be written as

$$\tilde{G} + \tilde{\mu}$$

where $\tilde{G}$ is the sum of all the G-wards maps and

$$\tilde{\mu} = \sum_{i,j,k} \mathrm{Id}^{\otimes i} \otimes \mu_j \otimes \mathrm{Id}^{\otimes k}$$

The component of the square of this map which goes to the terminal vertex of $\widehat{G_x}$ is

$$\tilde{G} \circ \tilde{\mu} + \tilde{\mu} \circ \tilde{G} = \sum_i g_i \circ \tilde{\mu} + \mu_r \left(\sum_{i_1+\cdots+i_r=n} g_{i_1} \otimes \cdots \otimes g_{i_r} \right)$$

where

$$\tilde{G} = \sum_{j_1+\cdots+j_q=k} g_{j_1} \otimes \cdots \otimes g_{j_q}$$

by Definition 5.14 and Lemma 5.11, part 6.

Theorem 5.18. *Proposition 5.13 and the construction above define a functor from the homotopy category of systems of hyperboxes to the homotopy category of A_∞-algebras.*

Composition of maps A_∞-algebras is, fittingly, only associative up to homotopy. The homotopy category is an honest category.

Proof. Proposition 5.13 and the discussion above the theorem define the maps on objects and morphisms. We need to prove the following:

Suppose that $\mathfrak{f}$ and $\mathfrak{g}$ are chain homotopic maps of systems of hyperboxes. Let J_s be the homotopy between F_s and G_s. Let $x \in C_s$ be a simple tensor. Define $j_n(x)$ to be the longest diagonal map on $\widehat{J_x}$ applied to x. These maps satisfy equation 8 by Definition 5.17 and Lemma 5.11. If x is inadmissible then equation 8 holds by the same argument as Proposition 5.13. Therefore f is A_∞-chain homotopic to g.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be maps of systems so that $\mathfrak{g}' \circ \mathfrak{g}$ is defined. The map of A_∞-algebras induced by $G' \circ G$ is A_∞-chain homotopic to $g' \circ g$ by from Corollary 5.8, Definition 5.14, and Lemma 5.11, part 6.

It is straightforward to check that the identity map of systems induces the identity map of A_∞-algebras. □

Remark 5.19. For the sake of concreteness, this section only discussed systems of hyperboxes in which summands are indexed by pairs of natural numbers, but of course one could repeat the construction over any set. In the next section we will work with pairs of crossingless matchings.

Remark 5.20. For inadmissible x we defined $\mu_k(x) = 0$ for lack of better options. It might seem more natural to say that multiplication is not even defined on x. This amounts to constructing an A_∞-*category* rather than A_∞-algebra. One ought to be able to adapt all the definitions of this section to build an A_∞-category from a system of hyperboxes in which the objects are natural numbers, $\mathrm{Hom}(i, j) = C_{i,j}$, and composition is given by the multiplication maps. Maps of systems induces A_∞-functors, and so on.

There is another way in which two systems of hyperboxes could yield homotopy equivalent A_∞-algebras. Here is what we have in mind: suppose that $f \simeq g$. It may hold that $f = f_1 \circ f_0$ and $g = g_1 \circ g_0$, but there is no way to fill in the diagram

diagram

So the compressions of these (one-dimensional) hyperboxes are homotopy equivalent, but the hyperboxes themselves are not.

Definition 5.21. Let $\mathcal{H}$ and $\mathcal{H}'$ be systems of hyperboxes over C. Suppose that these is an integer ℓ so that

- $C_s = C'_s$ for all s.
- If s does not contain ℓ, then $H_s = H'_s$.
- If s contains ℓ, then think of H_s and H'_s as a sequence of maps glued together along the dimension containing ℓ. The composition of these maps is chain homotopic by a homotopy so that
  - the homotopy vanishes on any tensor product whose underlying sequence does not contain ℓ.
  - the restriction of the homotopy to any subsequence agrees with the homotopy on that subsequence.

clarify, this should be like any of the other face conditions

We say that $\mathcal{H}$ and $\mathcal{H}'$ are *internally chain homotopic*.

Proposition 5.22. *If $\mathcal{H}$ and $\mathcal{H}'$ are internally homotopic, then $\tilde{\mathcal{A}}(\mathcal{H}) \simeq \tilde{\mathcal{A}}(\mathcal{H}')$.*

Proof. We will define a map f of algebras. Let x be a simple tensor of length k whose sequence s does not contain ℓ. Define

$$f_k(x) = \begin{cases} x & k = 1 \\ 0 & \text{otherwise} \end{cases}.$$

In other words, f looks like the identity map when $\ell \notin s$.

Now suppose that the last occurrence of ℓ in s is at the ith entry. The key observation is that $H_s^{k,\dots,i+1}$ and $H_s'^{k,\dots,i+1}$ are homotopic as maps of hyperboxes.

This follows from the functoriality of compression. So there is a map

this is the key point, improve a bit

$$F_s\colon\ H_s^{k,\dots,i+1} \to H_s'^{k,\dots,i+1}$$

whose length one edges are the identity. With the assumption that all the chain complexes are finite-dimensional, a standard argument shows that F_s has an inverse G_s up to homotopy. It follows that

$$H_s^{k,\dots,i+1} \simeq H_s'^{k,\dots,i+1}$$

as hyperboxes. Let f_s be the longest diagonal map in $\widehat{F_s}$. By our previous arguments and the second condition on the homotopy, the sum of all the f_s defines a map f of A_∞-algebras. Let g be the reverse map defined by hyperboxes G_s. We have $G_s \circ F_s \simeq \mathrm{Id}$. It follows that $g \circ f \simeq \mathrm{Id}$.

Now we check that $H_s^{k,\dots,i+1}$ and $H_s'^{k,\dots,i+1}$ satisfy the conditions of Definition 5.21. The first two conditions are clear.

□

6. A trisection class invariant

Definition 6.1. Let $\mathbf{t}$ be a tri-plane diagram. We say that $\mathbf{t}$ is in *plat form* if all three tangles are planar isotopic to half-plat closures of braids.

Every $(n, 0)$-tangle is isotopic (but not necessarily planar isotopic) to a half-plat closure. Therefore every trisection diagram may be put into plat form.

6.1. The system of hyperboxes.

Definition 6.2. Suppose that $\mathbf{t}$ is a tri-plane diagram in plat form. The *canonical surgery arcs* on

$$t_i\bar{t}_j \coprod t_j\bar{t}_k$$

are the arcs connecting the plats in $\bar{t}_j$ and t_j, oriented towards t_j. Number these arcs with 1 to n from top to bottom. For a diagram of the form

$$t_{i_1}\bar{t}_{i_2} \coprod t_{i_2}\bar{t}_{i_3} \coprod \cdots \coprod t_{i_k}\bar{t}_{i_{k+1}}$$

There are $(k-2)$ *families of canonical surgery arcs*, defined similarly. They are the red, dotted arcs in Figure 11.

Let $\mathbf{t} = (t_1, t_2, t_3)$ be a bridge trisection diagram in plat form. Let $s = (s_1, \dots, s_{k+1})$ be a sequence of length $k+1 \geq 2$ in $\{1, 2, 3\}$. Define

$$\mathcal{D}_s = (t_{s_1}\bar{t}_{s_2}) \coprod (t_{s_2}\bar{t}_{s_3}) \coprod \cdots \coprod (t_{s_{k-1}}\bar{t}_{s_{k+1}}).$$

For simplicity, let us first consider a tri-plane diagram in plat form with no crossings. (This implies that each tangle is the plat closure of the identity braid, but we won't

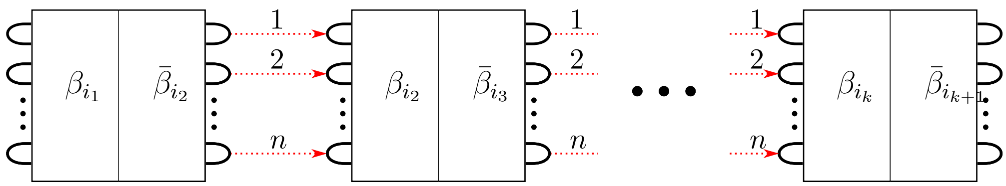

FIGURE 11

use that fact.) If $|s| = 2$, then set $H_s = \mathrm{CSz}(\mathcal{D}_s)$. If $|s| > 2$, then there are $k-2$ families of canonical surgery arcs on $\mathcal{D}_s$, each with n arcs. For each coordinate

$$\delta = (d_1, \dots, d_{k-2}) \in [0, n]^{k-2}$$

there is a diagram $\mathcal{D}_{s,\delta}$ given by performing surgery along the first d_i arcs in the i-th family. For example, $\mathcal{D}_{s,(0,\dots,0)} = \mathcal{D}_s$ and $\mathcal{D}_{s,(1,\dots,1)} = t_{s_1}\bar{t}_{s_k}$. Let

$$C_s = \bigoplus_{\epsilon \in [0,n]^{k-2}} \mathrm{CSz}(\mathcal{D}_{s,\delta}).$$

Fix a coordinate δ. A direction vector ϵ from δ picks out $\|\epsilon\|$ canonical arcs: for each $\epsilon_i = 1$, take the $(\delta_i + 1)$-st arc from the ith collection. In other words, use the "next" arcs in each axis with a 1.

Let $\mathcal{C}_{s,\delta,\epsilon}$ be the configuration whose underlying diagram is $\mathcal{D}_{s,\delta}$ and whose decorations are the arcs picked out by ϵ. Set

$$\begin{aligned} &D_\delta^\epsilon \colon \ \mathrm{CSz}(\mathcal{D}_{s,\delta}) \to \mathrm{CSz}(\mathcal{D}_{s,\delta+\epsilon}) \\ &D_\delta^\epsilon = \mathfrak{S}_{\mathcal{C}_{s,\delta,\epsilon}} \\ &D_s = \sum_{\delta,\epsilon} D_\delta^\epsilon. \end{aligned}$$

The (δ, ϵ)-cube in C_s is the Szabó complex of the link given by replacing each decoration from ϵ with a positive crossing. So $H_s = (C_s, D_s)$ is actually a hyperbox of chain complexes.

When $\mathbf{t}$ has crossings, there are two additional complications. First, the diagram $\mathcal{D}_{s_{\epsilon_1}}$ is isotopic to $s_1\overline{s_n}$ but not equal to it. This isotopy may always be realized by a sequence of Reidemeister 2 moves which cancel inverse Artin generators. For a fixed braid word β_i order these from the inside out: if $\beta = \sigma_{i_1} \cdots \sigma_{i_j}$, then

$$\beta^{-1}\beta = \sigma_{i_j}^{-1} \cdots \sigma_{i_1}^{-1} \sigma_{i_1} \cdots \sigma_{i_j}.$$

and the first cancellation is between $\sigma_{i_1}^{-1}$ and σ_{i_1}.

Suppose that the braid word underlying s_i has length ℓ_i. Write ℓ_i for the length of the braid word underlying s_i. Form a $(k-1)$-dimensional hyperbox of link diagrams of size

$$(n + \ell_{i_1}, \dots, n + \ell_{i_{k-1}}).$$

For each coordinate $\delta = (d_1, \dots, d_{k-1})$ there is a diagram $\mathcal{D}_{s,\delta}$ given as follows: if $d_i \leq n$, then perform surgery along the first e_i canonical arcs between $\overline{s_{i+1}}$ and s_{i+2}, just as in the crossingless case. If $e_i = n + m_i$ with $\ell_i \geq m_i > 0$, then perform all n surgeries, then perform the first m_i Reidemeister 2 moves on the result. $\mathcal{D}_{s,\delta}$ is the result of all these alterations to $\mathcal{D}$. Set

$$C_s = \bigoplus_{\epsilon \in (d_1 + c_1, \dots, d_n + c_n)^{k-2}} \mathrm{CSz}(\mathcal{D}_{s,\epsilon}).$$

Suppose that all of the coordinates of δ are less than n. The (δ, ϵ)-cube defines a diagram $\mathcal{D}_{\delta,\epsilon}$ by replacing the definitions with positive crossings. Define the edge maps on the (δ, ϵ)-cube so that the cube is the CSz complex of $\mathcal{D}_{\delta,\epsilon}$. This is the second complication: the edge maps include not just the decorations picked out by ϵ, but any number of the decorations from the crossings of $t_{s_i}\bar{t}_{s_{i+1}}$.

If δ has some coordinates greater than or equal to n, then ϵ may pick out Reidemeister 2 moves in addition to arcs. The map assigned to a Reidemeister 2 move on CSz is a sum of many configuration maps, see Section **??**. The map D_δ^ϵ is given by adding superimposing these configuration maps and surgery arcs.

did we actually say this? easy enough to add
improve

Proposition 6.3. *Let* $\mathbf{t}$ *be a tri-plane diagram in plat form. The recipe above defines a system of hyperboxes of chain complexes* $\mathcal{H}(\mathbf{t})$ *over*

don't think this is the right vocab

$$C = \bigoplus_{i,j=1}^{3} \mathrm{CSz}(t_i \bar{t}_j).$$

Proof. First we show that H_s is actually a hyperbox for any s. To do so, we show that the (δ, ϵ)-cube is a chain complex. If ϵ does not pick out any Reidemeister 2 moves then the cube is a CSz complex by definition.

So suppose that ϵ picks out some Reidemeister 2 moves. These moves have disjoint support, and therefore the maps associated to the commute up to homotopy. In [7] we showed that this homotopy is precisely described by the diagonals in the (δ, ϵ)-cube. Therefore this cube is a chain complex.

improve some

Now let's check that the assignment $s \mapsto H_s$ defines a system. It is clear that the construction satisfies the first condition. The second follows from the extension rule and the disconnected rule. Let F be the face of H_s between the ϵ and ϵ' corners. The extension rule implies that the map assigned to a handle attachment along a canonical surgery arc acts as the identity on the fixed sequence of s. Let c and c' be distinct subsequences in $c(\epsilon, \epsilon')$. Let γ and γ' be canonical surgery arcs which are attached as part of c and c', respectively. A configuration in F involving both γ

and γ' can never be connected. This implies that H_s satisfies the face condition, cf. Lemma 5.10, part (2). □

Definition 6.4. Let $\tilde{\mathcal{A}}(\mathbf{t})$ be the A_∞-algebra over $\mathbb{F}[W]$ constructed from Proposition 5.13, Proposition 6.3, and the discussion above. The underlying group is

$$\bigoplus_{i,j=1}^{3} \mathrm{CSz}(t_i\overline{t_j}).$$

For $y \in C_s$, $\mu_k(y)$ is the image of y under the longest diagonal map of $\widehat{H_s}$ applied to y.

The argument above applies to any conic, strong Khovanov-Floer theory with a vanishing differential on flat diagrams. One can rephrase the construction of H_s in terms of iterated mapping cones of one-handle attachment maps and Reidemeister 2 moves, and the argument of Proposition 6.3 holds word for word. The only part of argument which uses the language of Szabó homology is the extension rule, which holds up to homotopy for any strong Khovanov-Floer theory. If the internal differential vanishes, it holds on the nose. The upshot is that we can construct an A_∞-algebra with underlying vector space

$$\bigoplus_{i,j=1}^{3} \mathrm{CSz}(t_i\bar{t}_j)$$

and with multiplications given as above even for the Heegaard Floer homology of branched double covers (with the right choice of Heegaard diagrams.)

6.2. **The simplest example.** Let us compute $\tilde{\mathcal{A}}(\mathbf{t})$ in the case that $\mathbf{t}$ is the flat, bridge number 1 tri-plane diagram for the unknotted sphere in S^4. We call this algebra $\mathcal{I}$. Each summand of $\tilde{\mathcal{A}}(\mathbf{t})$ has rank two and $\mu_1 = 0$.

Let x be a simple tensor of length k. There is a hyperbox H_x underlying $\mu_k(x)$. The active part of any connected configuration which appears in H_x consists of some circles connected to each other in a line. The map assigned to such a configuration is zero (as long as the dimension is greater than one). The map assigned to a disconnected configuration is zero. We conclude that there are no non-zero configurations of dimension greater than one, and therefore all diagonal $\mathfrak{S}$ maps in H_x are zero. It follows that $\mu_k = 0$ for $k > 2$.

Suppose that $\mathbf{t}$ is the disjoint union of n copies of the 1-bridge unknot. The argument above, along with the Disconnected Rule for Szabó homology, implies that the higher Szabó maps all vanish. It is interesting to note that these configurations are trees and therefore have non-trivial maps under $\mathfrak{B}$.

7. Invariance of the algebra

Our goal in this section is to prove the following.

Theorem 7.1. *Let Σ be bridge-trisected surface in S^4. Let $\mathbf{t}$ be a tri-plane diagram in plat form for Σ. The A_∞-chain homotopy type of $\tilde{\mathcal{A}}(\mathbf{t})$ is an invariant of the trisection class of $\mathbf{t}$.*

Proof. We will prove something slightly stronger: let $\mathbf{t}'$ be a tri-plane diagram, not necessarily in plat form, for Σ. Let $\mathbf{t}$ be a Reidemeister-equivalent diagram in plat form. In the remainder of this section, we show that $\tilde{\mathcal{A}}(\mathbf{t})$ is invariant under

- braid isotopies (Proposition 7.2).
- Hilden moves (Proposition 7.3).
- interior Reidemeister moves and bridge sphere transpositions (Proposition 7.4).

The upshot is that one may think of the invariant as the assignment $\mathbf{t}' \mapsto \tilde{\mathcal{A}}(\mathbf{t})$, even if $\mathbf{t}'$ is not in plat form.

Observe that, by restricting to plat form, the only interior Reidemeister moves to worry about are Hilden moves and braid isotopies. But of course those are covered by the first two bullets. So in the third, we only consider bridge sphere transpositions. □

Proposition 7.2. *Let β and β' be braid words which represent equal elements of B_{2b}. Let $\mathbf{t} = (\breve{\beta}_1, \breve{\beta}_2, \breve{\beta}_3)$ and $\mathbf{t}' = (\breve{\beta}'_1, \breve{\beta}_2, \breve{\beta}_3)$. $\tilde{\mathcal{A}}(\mathbf{t})$ is chain homotopy equivalent to $\tilde{\mathcal{A}}(\mathbf{t}')$.*

Proof. It suffices to prove the theorem in the case that β and β' differ by a single braid commutation, braided Reidemeister 2 move, or triple-point move.

We begin with braid commutation. Consider the systems $\mathcal{H}(\mathbf{t})$ and $\mathcal{H}(\mathbf{t}')$. If s does not include 1, then $H_s = H'_s$. If it does, then the maps on corresponding edges are identical except that the order of the two Reidemeister moves has been swapped. Proposition 4.3 implies that $\mathcal{H}(\mathbf{t})$ and $\mathcal{H}(\mathbf{t}')$ are internally homotopic. (Rather than applying Proposition 3.3, one can just write down, in terms of cobordisms, a homotopy which shows that the order of two disjoint Reidemeister 2 moves doesn't matter.) Proposition 5.22 implies that $\tilde{\mathcal{A}}(\mathbf{t}) \simeq \tilde{\mathcal{A}}(\mathbf{t})$.

Now suppose that β and β' differ by a single braided Reidemeister 2 move. Without loss of generality suppose that β' has two more crossings than β, and call the added crossings *new*. Let s be a sequence which contains 1. Then $H_{s'}$ is larger than H_s: any axis which corresponds to 1 is two units longer than the same axis in H_s because it has to undo the new crossings. The other segments of $H_{s'}$ all correspond to segments of H_s in an obvious way. Add elementary extensions to H_s so that it has the same shape as $H_{s'}$ and so that corresponding pieces line up. Call the extended box H_s.

Define a map $\rho_s\colon H_s \to H'_s$ cube-by-cube as follows. Let C and C' be corresponding cubes of H_s and H'_s. Then either

- Neither C nor C' involves a undoing any crossings. In other words, C and C' differ only by one or zero Reidemeister 2 moves. Away from the supports of these moves the cobordisms in the cubes are identical. In other words, C and C' are Szabó complexes of link diagrams which differ by some Reidmeiester 2 moves. Define $\rho_s|_C$ to be the composition of all the Reidemeister 2 maps.
- C and C' do involve undoing some crossings, but not the new ones. Then C and C' are (iterated) mapping cones of Reidemeister 2 maps on links which differ by a Reidemeister 2 move. Define $\rho_s|_C$ to be the cone of the new Reidemeister 2 map on the mapping cone. (In other words, iterate the mapping cone again).
- C' undoes one of the new crossings. Then define $\rho_s|_C$ by the schematic in Figure 12.

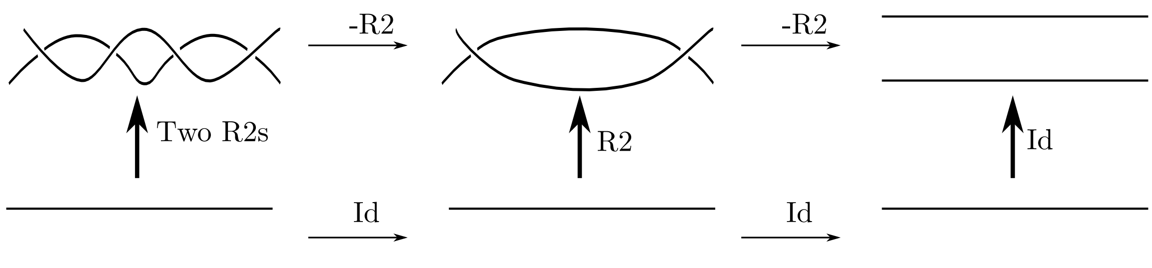

FIGURE 12

Figure 12 deserves a little more discussion. The vertical direction is the mapping direction. The horizontal axis should be thought of as a horizontal axis which corresponds to a 1 in s. Each horizontal arrow represents a map of cubical chain complexes, for example a Reidemeister 2 map on the Szabó chain complex of some link. The vertical arrows are defined in the same way. The maps assigned to the moves in Figure 12 commute up to homotopy because they commute cubewise. These homotopies are the dotted arrows. They are also defined via handle attachments and therefore can be extended to the Szabó complex of any link diagram. It follows that the schematic defines a map of hyperboxes.

Define ρ'_s by reversing all the Reidemeister 2 maps in bullet points above. We must show that ρ_s and ρ'_s constitute maps of systems. The following principle will apply more generally. ρ_s is defined as the conglomeration of cobordism maps. (Remember that Reidemeister maps are cobordism maps.) Suppose that, on some cube C, ρ_s applies two Reidemeister 2 maps which are applied to disjoint links. By the disconnected rule, every configuration which uses one-handles from both maps is zero.

Therefore ρ_s splits as the box product of two cubes the two Reidemeister 2 moves applied separately. It follows that ρ_s defines a map of systems ρ and similarly for ρ'_s.

$\rho_s \circ \rho'_s \simeq \mathrm{Id}$ by a chain homotopy J_s. Let's construct this J_s more explicitly. First, if $\mathcal{D}_s$ is unchanged by the Reidemeister 2 moves, then ρ_s and ρ'_s are identity maps and $J_s = 0$. If s has a one-element contraction sequence, then define $J_s|_F$ on the cube C as the homotopy between $\rho_s|_C \circ \rho'_s|_C$ and Id_C.

If F has a larger contraction sequence, then define $J_s|_F$ by equation 10 with $F_{s''_i} = (\rho' \circ \rho)_{s''_i}$ and $G_{s''_{i+1}} = \mathrm{Id}_{s''_{i+1}}$. If J_s is actually a homotopy between $(\rho' \circ \rho)_s$ and Id_s, then it induces a homotopy between $\rho' \circ \rho$ and Id, completing the proof. So all we need to show is that J_s is a hyperbox with length one maps equal to the identity. Both parts are clear.

The triple point move is similar but with an additional wrinkle. Define ρ_s cube by cube as above. If s does not contain 1 then $\rho_s = \mathrm{Id}$. For other s, note that H_s and H'_s have the same size. Call the crossings affected by the triple point move *altered*. It is straightforward to define maps between cubes which do not undo altered crossings. The difficulty is that the Reidemeister 3 move shuffles the order of the altered crossings.

To correct for this, insert into H_s an elementary extension on each axis which involves the altered crossing right before the altered crossings. Consider the hyperbox H''_s in which the elementary extension is changed to a pair of Reidmeister 3 moves on the affected crossings and the maps after the extension are changed to agree with those of H'_s. By Proposition 4.3, $H''_s \simeq H_s$ and therefore the systems H_s and H''_s are internally homotopic.

Extend H'_s by the identity along each altered axis right before the altered crossings are undone. Construct a map $\rho \colon H'_s \to H''_s$ using the Reidemeister 2 recipe: if corresponding cubes C and C' do not involve undoing altered crossings, then define $\rho_s|_C$ using the Reidemeister 3 maps. For cubes which undo altered crossings, the diagrams actually agree (near those crossings) and so the map can be defined using the same recipe. In the extended region, the map is quite simple, see the two-dimensional schematic in Figure 13.

□

Proposition 7.3. *Suppose that $\breve{\beta}$ and $\breve{\beta}'$ differ by a Hilden move. Let $\mathbf{t} = (\breve{\beta}, \breve{\beta}_2, \breve{\beta}_3)$ and $\mathbf{t}' = (\breve{\beta}', \breve{\beta}_2, \breve{\beta}_3)$. $\tilde{\mathcal{A}}(\mathbf{t})$ is chain homotopic to $\tilde{\mathcal{A}}(\mathbf{t}')$.*

Proof. Suppose that β and β' are not equal as braids, but nevertheless $\breve{\beta}$ is isotopic to $\breve{\beta}'$. Lemma 2.1 implies that there is some $h \in H_n$ so that βh is isotopic to β'. There is a diagrammatic cobordism

$$\breve{\beta} \to \breve{\beta h}$$

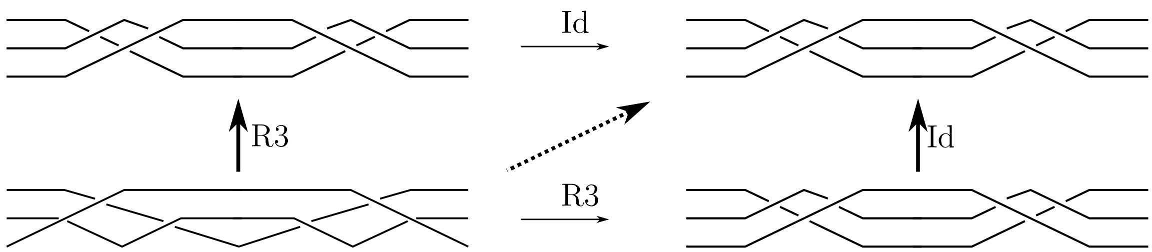

FIGURE 13. The schematic for the map ρ in the extension region. The top is H'' and the bottom is H'. In this case the square is commutative on the nose so no homotopy is necessary.

whose support is disjoint from β; this follows from the motion group interpretation of H^n. This cobordism is the composition of a sequence of Reidemeister moves, so it induces a chain homotopy equivalence on all the relevant Szabó chain groups.

It suffices to consider the case in which h is one of Tawn's generators. The support of each generator is a small neighborhood of the plats, so the crossings they add will always be canceled first. Write H' for the system underlying $\tilde{\mathcal{A}}(\mathbf{t}')$. The new crossings are the first to be canceled. The Reidemeister 2 moves have support disjoint from all the canonical surgery arcs (except for the ones which are pushed around by the Hilden move). Let H'' be the system which is identical to H' except that the new unwindings are moved to be right after those handle attachments. H'' is internally chain homotopic to H'.

Now extend H_s by the identity so it has the same shape as H''_s and so that the extensions lie over the cancellations of the new handles. We will refer to the result as H_s. We cook up a map for each of Tawn's generators. The map will always be the identity in the region "after" the new crossings have been canceled. The map will be a composition of Reidemeister moves which realize the Hilden move in the region "before" the relevant plats are connected. So the only challenge is to define the map in the region in which the relevant plats are connected and the new crossings are canceled.

t_i is the easiest. The downwards maps are the obvious Reidemeister maps. Consider the cobordism $\mathfrak{h} \circ r$, using the notation from Figure 14. This cobordism is isotopic to one which begins with a Reidemeister 2 move on (say) the left tangle, then connects the two plats. (This uses movie move 7 and 13 in Bar-Natan's reckoning [3].) The support of the Reidemeister 2 move is disjoint from the canonical surgery arc, so the maps assigned to those two cobordisms commute up to homotopy. This shows that $\mathfrak{h} \circ r \simeq r' \circ \mathfrak{h}$ in Bar-Natan's cobordism category. Therefore the homotopy can be chosen to be a map defined by handle attachments, and so Figure 14 defines a map of hyperboxes.

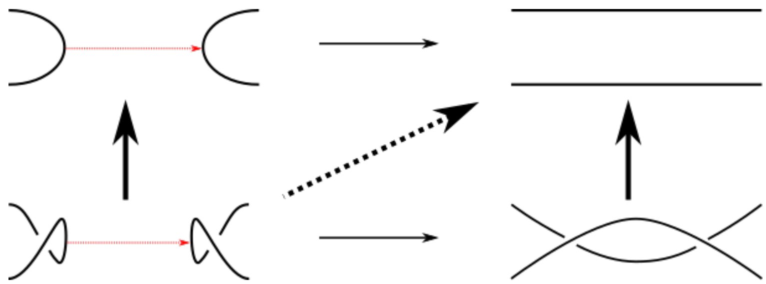

Figure 14. The interesting part of the map for t_i.

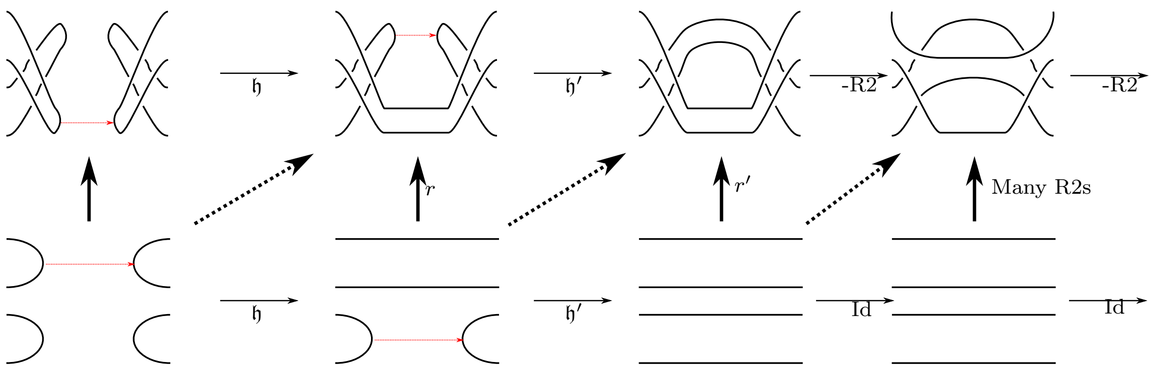

Figure 15

Figure 15 shows the argument for s_i. We have changed the order of handle attachment, but of course the two systems are internally homotopic. The first square commutes (up to homotopy) because the bottom plats can be passed under the upper plats by isotopies which are disjoint from $\mathfrak{h}$. For the second square, repeatedly use movie move 15 to show that $\mathfrak{h}' \circ r$ is equivalent to the cobordism in Figure 16. This cobordism is a composition of Reidemeister 2 moves with a disjoint canonical handle attachment. Swap the order of these two maps to obtain $r' \circ \mathfrak{h}'$. That the next square and all the ones after it commute up to homotopy follows from Proposition 4.3 (compare with the arguments in the previous proof.)

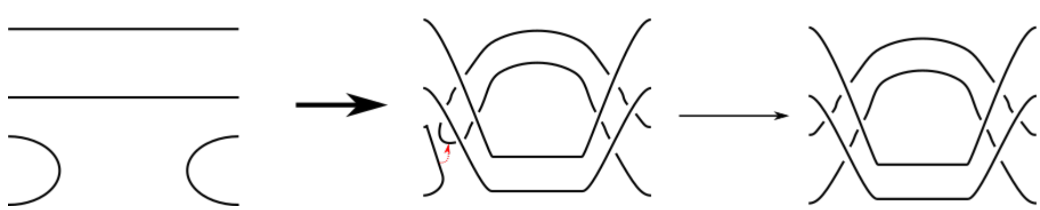

Figure 16

r_1 and r_2 are basically identical to s_i: use movie move 15 to write a cobordism as a sequence of Reidemeister 2 moves and disjoint canonical handle attachments.

These maps constitute maps of hypercubes because they are defined locally. They are invertible, up to homotopy, because all of their components are Reidemeister moves: one can reverse the maps and run the same argument. Apply the same process from the previous proof to obtain maps of systems. These homotopies are also given by cobordisms, so we can obtain homotopies between maps of systems. □

Proposition 7.4. *Suppose that* $\mathbf{t}$ *and* $\mathbf{t}'$ *are tri-plane diagrams in plat form which differ by a braid transposition. Then* $\tilde{\mathcal{A}}(\mathbf{t}) \simeq \tilde{\mathcal{A}}(\mathbf{t}')$.

The proof is essentially the same as that of Proposition **??**.

To define an invariant of $\mathcal{K}$ up to isotopy, we must understand the behavior of $q_i(\mathbf{t})$ under stabilization. To do so we must broaden the class of tri-plane diagrams to which our construction applies. Say that a $2b$-bridge tangle diagram is in *hybrid form* if it is a disjoint union of the plat closure of a $(2b-2k)$ braid and k crossingless arcs. A tri-plane diagram is in hybrid form if each of its tangles is.

Let $\mathbf{t}$ be in hybrid form. Define $\tilde{\mathcal{A}}(\mathbf{t})$ as above. There are two novelties. First, the ordering of the arcs is slightly different. Suppose that γ belongs to a braid plat and γ' to a crossingless plat. Then $\gamma > \gamma'$. If γ and γ' are both braid or both crossingless plats, say $\gamma > \gamma'$ exactly as before. Second, the braid crossings are unwound before crossingless plats are connected.

Proposition 7.5. *The construction of* $\tilde{\mathcal{A}}(\mathbf{t})$ *extends to diagrams in hybrid form. Suppose that* $\mathbf{t}$ *and* $\mathbf{t}'$ *are in the same trisection class and are in plat or hybrid form. Then* $\tilde{\mathcal{A}}(\mathbf{t}) \simeq \tilde{\mathcal{A}}(\mathbf{t}')$.

Proof. Let $\mathbf{t}$ be in hybrid form. That $\tilde{\mathcal{A}}(\mathbf{t})$ is an A_∞-algebra follows the same argument as above. So does invariance.

$\mathbf{t}$ is isotopic to a diagram $\mathbf{t}'$ in plat form. We may choose this isotopy so that it only moves the crossingless arcs of $\mathbf{t}$. For a diagram in plat form we may freely rearrange the order of the plat attachments. To see that $\tilde{\mathcal{A}}(\mathbf{t}) \simeq \tilde{\mathcal{A}}(\mathbf{t}')$, put a dot on the tip of each braid plat of $\mathbf{t}$ which becomes a crossingless plat in $\mathbf{t}'$. These dots mark the points which meet the surgery arc on that plat. Now choose an isotopy from $\mathbf{t}'$ to $\mathbf{t}$ which first slides the dot past other arcs. The dot traces out a path γ'. Call this intermediate diagram $\mathbf{t}''$. Define $\tilde{\mathcal{A}}(\mathbf{t}'')$ by attaching extending the surgery arc to the slid plat by γ. Use the arguments of Proposition **??** and movie move 15 to show that $\tilde{\mathcal{A}}(\mathbf{t}'') \simeq \tilde{\mathcal{A}}(\mathbf{t})$. Now slide the rest of the arc to obtain $\mathbf{t}'$. The argument of Proposition **??** shows that $\tilde{\mathcal{A}}(\mathbf{t}') \simeq \tilde{\mathcal{A}}(\mathbf{t}'')$.

□

Let $\mathbf{t}$ be the flat diagram of the bridge number one, unknotted sphere. Let $\mathbf{t}'$ be the stabilization shown in FIGURE. Then FIGURE shows that μ_3 does not vanish on $\mathcal{A}(\mathbf{t}')$. This shows that stabilization can dramatically change the character of $\mathcal{A}$. (Note that in any case the rank of $\mathcal{A}(\mathbf{t}')$ and its homology must be greater than that of $\mathcal{A}(\mathbf{t})$ and its homology, respectively.)

8. Units, the homology algebra, and minimal models

In this section we study some basic properties of $\tilde{\mathcal{A}}(\mathbf{t})$ and produce some equivalent algebras of lower rank.

8.1. **Units.** Let $\mathbf{t} = (t_1, t_2, t_3)$ be a tri-plane diagram in plat form. Observe that $\mathrm{CSz}(t_i\bar{t}_i)$ is an A_∞-subalgebra of $\tilde{\mathcal{A}}(\mathbf{t})$. Define

$$\mathcal{I}(\mathbf{t}) = \mathrm{CSz}(t_1\bar{t}_1) \oplus \mathrm{CSz}(t_2\bar{t}_2) \oplus \mathrm{CSz}(t_3\bar{t}_3)$$

$\mathcal{I}(\mathbf{t})$ is an A_∞-algebra. Observe that $t_1\bar{t}_1$ is isotopic to the plat closure of the identity braid entirely by Reidemeister 2 moves with support contained entirely in $\beta_1\bar{\beta}_1$.

Proposition 8.1. *The A_∞-algebra $\mathrm{CSz}(t_i\bar{t}_i)$ is A_∞-chain homotopy equivalent to $\mathcal{I}^n$.*

The proof is basically that of Proposition 7.4. Recall that $\mathcal{I}$ is the A_∞-algebra assigned to the genus 0 trisection of the unknotted sphere. In Section 6.2 we showed that the higher operations on $\mathcal{I}$ all vanish.

Definition 8.2. Define $\mathcal{A}(\mathbf{t})$ to be an A_∞-algebra on the vector space

$$\mathcal{I}_1^{\otimes n} \oplus \mathcal{I}_2^{\otimes n} \oplus \mathcal{I}_3^{\otimes n} \oplus \bigoplus_{i,j=1,2,3 i\neq j} \mathrm{CS}(t_i\bar{t}_j).$$

where $\mathcal{I}_i^{\otimes n}$ is identified with the algebra of the unwinding of $t_i\bar{t}_i$. The operations on $\mathcal{A}$, which we also call m_i, are defined identically to those on $\tilde{\mathcal{A}}$ with the addition of unwinding maps when the codomain is $\mathrm{CSz}(t_i\bar{t}_i)$.

Corollary 8.3. $\mathcal{A}(\mathbf{t})$ *is an A_∞-algebra, and its chain homotopy type is an invariant of the trisection class of* $\mathbf{t}$.

$\mathcal{A}(\mathbf{t})$ has substantially lower rank than $\tilde{\mathcal{A}}(\mathbf{t})$. Another advantage is that it has a unit. The unit in $\mathcal{I}^{\otimes n}$ is the diagram labeled with all pluses. Write $\iota \in \mathcal{A}(\mathbf{t})$ for the sum of units in $\mathcal{I}_1^{\otimes n}$, $\mathcal{I}_2^{\otimes n}$, and $\mathcal{I}_3^{\otimes n}$.

Lemma 8.4. *ι is a unit in* $\mathcal{A}(\mathbf{t})$.

Proof. Certainly $m_1(u) = 0$ and $m_2(u, x) = x$ for all $x \in \mathcal{A}(\mathbf{t})$. Let $y = x_1 \otimes \cdots \otimes x_n$ be a simple tensor so that $x_i = u$ for some i. We aim to show that $\mu_i(y) = 0$. We may assume without loss of generality that $y \in C_s$ for some sequence s.

Let H_y be the hyperbox used to compute $\mu_i(y)$. Consider a configuration in which one of the circles of x_i is active. That circle has either degree one or indegree and outdegree one. Every Szabó and Bar-Natan configuration map vanishes if that circle is labeled v_+.[4] It follows from the definition of compression that μ_i vanishes on y. □

8.2. **The homology algebra.** In this section we show that, for connected surfaces, the associative algebra $H(\mathcal{A}(\mathbf{t}))$ only detects the genus. We use Otal's theorem on bridge splittings of unlinks.

Theorem 8.5. *Let* $\mathbf{t}$ *be an* $(n; c_1, c_2, c_3)$ *triplane diagram in plat form for an oriented, connected surface* S*. The graded isomorphism type of the associative algebra* $H(A(\mathbf{t}))$ *is determined by* n*,* c_1*,* c_2*, and* c_3*.*

Proof. Write m for multiplication on $H(A(\mathbf{t}))$. As $t_i\bar{t}_j$ is an unlink, $\mathrm{Kh}(t_i\bar{t}_j)$ has a unique generator of highest quantum grading called Θ_{ij}. The set

$$\{\lambda\theta_{ij} : \lambda \in \Lambda \text{ a monomial}\}$$

is a basis for $\mathrm{Kh}(t_i\bar{t}_j)$. Let

$$m_{ijk} = m(\Theta_{ij} \otimes \Theta_{jk}).$$

There is a *basepoint action* on $\mathrm{CKh}(t_i\bar{t}_j)$, see [4] which satisfies

$$m(X_p\Theta_{ij} \otimes \Theta_{jk}) = X_p m(\Theta_{ij} \otimes \Theta_{jk})$$

as long p lies in t_i or $\bar{t}_k$. Choose $2b$ basepoints on each link where the two tangles meet. This turns each Khovanov homology group into a cyclic module over

$$\Lambda = \mathbb{F}[X_1, \ldots, X_{2b}]/(X_1^2, \ldots, X_{2b}^2).$$

m is a Λ-module map. Therefore m is determined by the values of m_{ijk}.

m_{ijk} can be described cobordism-theoretically: it is given by $c_{ij} + c_{jk}$ zero-handle attachments and then n one-handle attachments. Cap off this cobordism with two-handles to get a handle decomposition of $\mathcal{K}$. The cobordism is connected if and only if $\mathcal{K}$ is.

To compute m_{ijk}, one needs to describe this cobordism diagrammatically. It begins with the zero-handles and some Reidemeister moves to get $t_i\bar{t}_j \coprod t_j\bar{t}_k$. The Reidmiester maps are graded isomorphisms and will carry highest generators to highest generators.[5] Require that this sequence begin with Reidemeister 1 moves to put the circles into Otal position and that the rest of the sequence consists of Hilden moves. Choose a basepoint on each initial crossingless component. It follows from the functoriality of Khovanov homology that m_{ijk} is determined by the multiplication between the Otal links. These moves may cross the basepoints, but because all the diagrams are crossingless, the basepoint actions on homology commute with Reidemiester (and therefore Hilden) moves. So for any two tri-plane diagrams $\mathbf{t}$ and

[4]This is identical to the argument that the algebra $\mathcal{H}^n$ in [2] is unital.

[5]Note that this is not necessarily true for the other generators – Khovanov homology does not satisfy that much naturality!

$\mathbf{t}'$ with the same combinatorial data, there is an isomorphism $\Lambda \to \Lambda$ and an isomorphism of Λ-modules, twisted by this isomorphism, so that $H(A(\mathbf{t})) \simeq H(A(\mathbf{t}'))$. □

still not happy with this. write a map using the basepoints from the algebra to some kind of otal construction.

This theorem has precedents in independent work of Rasmussen [?] and Tanaka [?]. (It is somewhat surprising that the proof does not use their work.) There is a natural way to obtain an invariant of closed surfaces in S^4 from Khovanov homology: puncture the surface at its top and bottom, find a movie presentation Σ, determine the associated map

$$F_\Sigma\colon\ \mathrm{Kh}(U) \to \mathrm{Kh}(U),$$

and compute the Θ-coefficient of $F_\Sigma(U)$. Functoriality implies that this map does not depend on the particular movie presentation. For grading reasons this map must vanish if $\mathcal{K}$ is not a torus, and Rasmussen and Tanaka showed that the map takes the same value on any torus. Tanaka proved a similar statement for Bar-Natan's deformation.

Remark 8.6. Instead of studying $H(\mathcal{A}(\mathbf{t}))$, one could form a differential graded algebra $A(\mathbf{t})$ as in the previous section but using only the Khovanov differential. Observe that $H(A(\mathbf{t})) \cong H(\mathcal{A}(\mathbf{t}))$ as algebras. But Theorem 8.5 does not imply that $A(\mathbf{t})$ is determined by the genus of $\mathcal{K}$. It would be interesting to study Massey products on this algebra.[6]

Theorem 8.5 and the following theorem of Kadeishvili allow us to make $\mathcal{A}(\mathbf{t})$ more concrete as an invariant.

Theorem (Kadeishvili)**.** *Let A be an A_∞-algebra. There is an A_∞-structure on $H(A)$ so that $\mu_1 = 0$, $\mu_2 = m_2^*$, and A is A_∞-quasi-isomorphic to A. If A is unital, then the structure and quasi-isomorphism may be chosen to be unital as well.*

An A_∞-algebra with $\mu_1 = 0$ is called *minimal*. It follows that there is a (non-unique) minimal model for $\mathcal{A}(\mathbf{t})$ with rank $2^{3c} + 2^{3n}$. For any two tri-plane diagrams with the same combinatorial data, we obtain two minimal A_∞-algebras which are isomorphic as associative algebras: the invariant is an A_∞-structure on this algebra.

9. Massey products and an isotopy class invariant

In Section 6 we applied CSz to a system of hyperboxes to obtain an A_∞-algebra. In this section we include Bar-Natan's perturbation to obtain an isotopy class invariant. Our presentation is somewhat roundabout: first, we consider how the ideas of Section 6 apply to the full differential $d_{\mathrm{Sz}} + d_{\mathrm{BN}}$. We study the behavior of the top degree generators under a triple product map. Then we show that the triple product

[6] Massey products and A_∞-constructions are closely related, see [?], so in some sense this is the subject of Section 6. But they are not identical.

is well-defined even if one throws out the Bar-Natan contributions. Finally, we show that this product is, in some sense, invariant under stabilization. There is surely a faster route to defining the final invariant, but the structure of the intermediate algebra is interesting in its own right.

The recipe of Definition 6.4 is not compatible with CS. Let μ'_i be the putative A_∞ operations. Even the A_2-relation does not hold:

$$\mu'_2(\mu'_1(x) \otimes y) + \mu'_2(x \otimes \mu'_1(y)) + \mu'_1\mu'_2(x \otimes y) = U^{-1}W\mu'_2(d_{\mathrm{BN}}x \otimes d_{\mathrm{BN}}).$$

because of the failure of the Künneth formula for Bar-Natan's theory. The abberant term on the left is the part of $d \circ d_{\mathrm{BN}}$ in which d_{BN} counts Bar-Natan configurations with support on both sides of the canonical arcs but not the canonical surgery arcs. In terms of hyperboxes, the alleged system will fail the face condition: a one-handle attachment inside the contraction sequence will have an effect on the fixed sequence.

unclarity here strongly argues to move Bar-Natan's deformation to here. Also makes the beginning of the paper more standard.

Nevertheless, there is are map μ'_i defined via not-quite-systems of hyperboxes. We will exploit μ'_1, μ'_2, and μ'_3 to define an isotopy class invariant. Let $\aleph_3$ be the left side of equation 6. Then the μ' maps satisfy

$$\begin{aligned}\aleph_3 = U^{-1}W\big(&\mu'_3(d_{\mathrm{BN}}(x) \otimes d_{\mathrm{BN}}(y) \otimes z) \\ &+ \mu'_3(d_{\mathrm{BN}}(x) \otimes y \otimes d_{\mathrm{BN}}(y)) + \mu'_3(x \otimes d_{\mathrm{BN}}(y) \otimes d_{\mathrm{BN}}(z))\big) \\ &+ U^{-2}W^2\mu'_3(d_{\mathrm{BN}}(x) \otimes d_{\mathrm{BN}}(y) \otimes d_{\mathrm{BN}}(z)).\end{aligned}$$

Call this right side $\daleth_3$. Observe that $\daleth_3$ is divisible by W.

9.1. **Top generators.** Let $\mathbf{t}$ be an $(n; c_{12}, c_{23}, c_{31})$ tri-plane diagram for $\mathcal{K}$. $\mathrm{HS}(t_i\bar{t}_j)$ has a unique element Θ_{ij} with greatest quantum grading. Write θ_{ij} for a homogeneous cycle representative of Θ_{ij}. Note that θ_{ii} is unambiguous.

where do we use this

Lemma 9.1. *Let $x \in \mathrm{CS}(t_i\bar{t}_j)$ be a canonical generator such that $dx \neq 0$. Then $\mu'_2(x \otimes \theta_{jk})$ cannot have coefficient at θ_{ik} which is divisible by W but not W^2.*

Proof. For x to have such a term, it must have homological degree (-1) – otherwise the power of W would be larger. Setting H to 1, the degree one part of

$$\langle \mu'_2(x \otimes \theta_{jk}), \theta_{ik} \rangle \tag{11}$$

essentially counts certain two-dimensional configurations in which one decoration comes from a crossing and the other comes from a handle attachment from a canonical surgery arc. We may assume that the circles which meet the surgery arc are v_+-labeled. (Any other sort of canonical generator cannot contribute to the coefficient of θ_{ik} by the filtration rule.)

Observe that

$$d\mu'_2(x \otimes \theta) = \mu'_2(dx \otimes \theta) + \mu'_2(x \otimes d\theta) + H^{-1}W\mu'_2(d_{\mathrm{BN}}x \otimes d_{\mathrm{BN}}\theta).$$

$d\theta = 0$ and $d_{BN}\theta$ cannot contribute to the expression in (11) because of its homological degree. It follows that

$$\langle d\mu_2'(x \otimes \theta), \theta_{ik} \rangle = \langle \mu_2'(dx \otimes \theta), \theta_{ik} \rangle$$

In fact the term on the left is zero. For suppose that $d\mu_2'(x \otimes \theta) = a\theta_{jk} + y$ with $a \in \mathbb{F}[U, W]$. It follows that y is also a cycle. If y is a boundary then $[\theta_{jk}] = 0$. If y is not a boundary, then it represents a class with lower grading. But then $a[\theta_{jk}] = [y]$, which is impossible. It follows that

$$\langle \mu_2'(dx \otimes \theta_{jk}), \theta_{ik} \rangle = 0$$

Now suppose that c is a crossing which supports a configuration in the sense above. Write d_c for the component of d which involves c. We can think of

$$\langle \mu_2'(d_c x \otimes \theta), \theta_{ik} \rangle$$

as a count of two-dimensional configurations which have been divided into two one-dimensional configurations.[7] Let $\mathcal{C}$ be a two-dimensional configuration which uses c and contributes to

$$\langle \mu_2'(x \otimes \theta_{jk}), \theta_{ik} \rangle.$$

If $\mathcal{C}$ is a Szabó configuration, then it must be connected. It follows from the filtration rule that all the active circles must be positive. Therefore the configuration is either of type 1 or 8 in Szabó's list, page WHICH of [9]. Write $\mathcal{C}_c$ and $\mathcal{C}'$ for the one-dimensional configurations which make up $\mathcal{C}$. One can check directly that, if $\mathcal{C}$ is of one of these two types, then

$$U^{-1}W \langle \mu_2'(d_c(x) \otimes \theta_{ik}), \theta_{ik} \rangle = \langle F_{\mathcal{C}}(x \otimes \theta_{ik}), \theta_{ik} \rangle$$

where $F_{\mathcal{C}}$ is the part of μ_2' which includes $\mathcal{C}$. The same holds for Bar-Natan configurations by the principle at the top of the section. It follows that

$$\langle \mu_2'(x \otimes \theta_{ik}), \theta_{ik} \rangle = U^{-1}W \langle \mu_2'(dx \otimes \theta_{ik}), \theta_{ik} \rangle = 0$$

□

this still seems slightly off – what if their are multiple configurations for a crossing?

Lemma 9.2. *Fix representatives θ_{ij}, θ_{jk}, and θ_{ki}. If $\mu_2'(\theta_{ij} \otimes \theta_{jk})$ has a non-trivial coefficient at θ_{ik}, then that coefficient is*

$$H^{\frac{n + c_{ik} - c_{ij} - c_{jk}}{2}}.$$

Proof. From consideration of the homological degree we can think of μ_2 instead of μ_2'. The naïve quantum degree of μ_2' is $-n$. The grading of $\theta_{ij} \otimes \theta_{jk}$ is $c_{ij} + c_{jk}$, so $\mu_2'(\theta_{ij} \otimes \theta_{jk})$ has grading $c_{ij} + c_{jk} - n$. If H^a is the coefficient of θ_{ik} in $\mu_2'(\theta_{ij} \otimes \theta_{jk})$, then

$$-a_{ijk} + c_{ik} = c_{ij} + c_{jk} - n$$

[7]Compare to "broken polygons" in, for example, Heegaard Floer homology.

and therefore

$$a = \frac{n + c_{ik} - c_{ij} - c_{jk}}{2}.$$

□

For crossingless links, the formula above can be worked out by counting the number of merges and splits in Ξ. It's a fun exercise to check that the formulas coincide.

The formula implies that $\mu_2(\theta_{ij} \otimes \theta_{jk})$ has coefficient 0 at θ_{ik} if $n + c_{ij} - c_{jk} + c_{ki}$ is odd. One can check this directly for the standard diagrams of projective planes. If i, j, and k are distinct then

$$a_{ijk} = -\frac{\chi(\mathcal{K})}{2} + c_{ik}.$$

So the coefficient vanishes if $\chi(\mathcal{K})$ is odd. Note also that

$$a_{iji} = \frac{n + n - 2c_{ij}}{2} = n - c_{ij}$$

and

$$a_{iii} = 0.$$

ok but why subtract one?

Lemma 9.3. *The $H^{n-\chi(\mathcal{K})/2-1}W\theta_{ii}$-coordinate of $\mu_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})$ does not depend on the choice of homogeneous representatives.*

Proof. There is only one choice for θ_{ii}. Let dx be a boundary in $\mathrm{CS}(t_i\bar{t}_j)$ with homological degree 0. Then

$$\mu_3'(dx \otimes \theta_{jk} \otimes \theta_{ki}) = \mu_2'(x \otimes \mu_2'(\theta_{jk} \otimes \theta_{ki})) + \mu_2'(\mu_2'(x \otimes \theta_{jk}) \otimes \theta_{ki}) + \beth_3(x, \theta_{jk}, \theta_{ki}).$$

Every term of $\beth_3$ involves applying d_{BN} to at least one θ and therefore it cannot contribute. Lemma 9.1 shows that $\mu_3'(dx \otimes \theta_{jk} \otimes \theta_{ki})$ has coefficient zero at $H^{n-\chi(\mathcal{K})/2}W\theta_{ii}$. A similar proof applies with dy or dz in the place of θ_{jk} or θ_{ki}. □

9.2. **Triple product invariants.**

Definition 9.4. Let $q'_{ijk}(\mathbf{t}) \in \mathbb{Z}/2\mathbb{Z}$ to be the coefficient of $U^{n-\chi(\mathcal{K})/2-1}W\theta_{ii}$ in

$$\mu_3'(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}).$$

When the values of i, j, and k are not important we will omit them. Lemma 9.3 shows that $q_{ijk}(\mathbf{t})$ does not depend on the choice of θs. We chose the exponent in light of Lemma 9.2.

a bit more

Proposition 9.5. *$q'(\mathbf{t})$ does not depend on the choice of representatives θ.*

Proof. Follows immediately from Lemma 9.3. □

Observe also that, in computing q', one can almost forget that the θs come from diagrams with crossings: no configuration involving a crossing can contribute to θ_{ii}. In other words, one can treat the θs as sums of generators from flat diagrams. One still has to remember the crossings to unwind them at the end, but these unwinding maps cannot involve any higher configurations from the same reason. Therefore we may compute q by a hyperbox of chain complexes in which each link diagram is crossingless.

Write H for such a hyperbox underlying the computation of $\mu_3'(x)$ for some simple tensor x which is a summand of a θ. Write $\bar{H}$ for the same hyperbox but with all the higher Bar-Natan differentials removed. This is still a hyperbox because each cube of H' belongs to a link diagram with at most two crossings, so the two-dimensional differential is totally unconstrained.

Definition 9.6. Let $q_{ijk}(\mathbf{t}) \in \mathbb{Z}/2\mathbb{Z}$ to be the coefficient of $U^{n-\chi(\mathcal{K})/2-1}W\theta_{ii}$ in

$$\bar{\mu}_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}).$$

where $\bar{\mu}_3$ has the same underlying hyperbox as μ_3' but without any higher Bar-Natan differentials.

Proposition 9.7. *$q(\mathbf{t})$ does not depend on the chosen θs.*

Before proving the proposition, note that $d_{\mathrm{Kh}} + d_{\mathrm{BN}}$ is a differential. Write μ_3'' for the resulting triple multiplication. Then

$$\mu_3' = \mu_3'' + \bar{\mu}_3$$

Observe that $(d_{\mathrm{Kh}} + d_{\mathrm{BN}}, \mu_2'', \mu_3'')$ forms the same sort of "perturbed A_3-algebra" as (μ_1', μ_2', μ_3'); after all, the Bar-Natan differential is the source of the perturbation.

Proof. We claim that

$$\begin{aligned}
\mu_3''(\partial x \otimes \theta_{jk} \otimes \theta_{ki}) &= \mu_3''((d_{\mathrm{Kh}} + d_{\mathrm{BN}})x \otimes \theta_{jk} \otimes \theta_{ki}) + \mu_3(d_{\mathrm{Sz}}x \otimes \theta_{jk} \otimes \theta_{ki}) \\
&= \mu_2''(x \otimes \mu_2''(\theta_{jk} \otimes \theta_{ki})) + \mu_2''(\mu_2''(x \otimes \theta_{jk}) \otimes \theta_{ki}) \\
&\quad + \mu_3''(x \otimes (d_{\mathrm{Kh}} + d_{\mathrm{BN}})\theta_{jk} \otimes \theta_{ki}) + \mu_3''(x \otimes \theta_{jk} \otimes (d_{\mathrm{Kh}} + d_{\mathrm{BN}})\theta_{ki})) \\
&\quad + \mu_3''(d_{\mathrm{Sz}}x \otimes \theta_{jk} \otimes \theta_{ki}) \\
&\quad + \beth_3'(x, \theta_{jk}, \theta_{ki}).
\end{aligned}$$

This is just the perturbed A_3-relation. Note that $d_{\mathrm{Kh}} + d_{\mathrm{BN}}\theta \neq 0$ because the θs are ∂-cycles – Lemma 9.1 does not apply for this reason. Nevertheless, $(d_{\mathrm{Kh}} + d_{\mathrm{BN}})\theta$ is divisible by W. One way to see this is that

$$(d_{\mathrm{Kh}} + d_{\mathrm{BN}})\theta = d_{\mathrm{Sz}}\theta.$$

It follows that the terms in third and fourth lines cannot contribute to the W-degree one coefficient of θ_{ii}. Neither can the $\daleth'_3$ term for the usual reasons. Therefore, writing $\langle -, - \rangle$ for the W-degree one components,

$$\langle \mu''_3(\partial x \otimes \theta_{jk} \otimes \theta_{ki}), \theta_{ii} \rangle = \langle \mu''_2(x \otimes \mu''_2(\theta_{jk} \otimes \theta_{ki})) + \mu''_2(\mu''_2(x \otimes \theta_{jk}) \otimes \theta_{ki}), \theta_{ii} \rangle$$

Now the arguments of Lemmas 9.1 and 9.3 does apply, and therefore the W-degree one part of

$$\langle \mu''_3(\partial x \otimes \theta_{jk} \otimes \theta_{ki}), \theta_{ii} \rangle$$

vanishes. A similar argument applies to adding a cycle in the second or third position.

It follows that the terms in the second line are divisible by W^2 and therefore do not contribute to $q(\mathbf{t})$. Therefore the W-degree one part of

$$\langle \mu''_3(\partial x \otimes \theta_{jk} \otimes \theta_{ki}), \theta_{ii} \rangle$$

is zero. Therefore the W-degree one part of

$$\langle \bar{\mu}_3(\partial x \otimes \theta_{jk} \otimes \theta_{ki}), \theta_{ii} \rangle = \langle (\mu'_3 + \mu''_3)(\partial x \otimes \theta_{jk} \otimes \theta_{ki}), \theta_{ii} \rangle$$

is zero. It follows that $p(\mathbf{t})$ is well-defined. □

Proposition 9.8. *Suppose that* $\mathbf{t}$ *and* $\mathbf{t}'$ *belong to the same trisection class. Then* $q_{ijk}(\mathbf{t}) = q_{ijk}(\mathbf{t}')$ *for all* i, j, *and* k, *and similarly for* q'.

Proof. In Section 7 we constructed a chain homotopy equivalence of A_∞-algebras

$$\rho\colon \mathcal{A}(\mathbf{t}) \to \mathcal{A}(\mathbf{t}').$$

It satisfies $\rho_{1,*}(\Theta_{ij}) = \Theta'_{ij}$ for all i and j. Using the same arguments, we can produce linear maps

$$\rho'_i\colon \mathcal{A}(\mathbf{t})^{\otimes i} \to \mathcal{A}$$

using the same cobordisms and CS. ρ'_1 is a graded chain homotopy equivalence and therefore $\rho_{1,*}(\Theta_{ij}) = \Theta'_{ij}$. The modified A_2-relation is

$$\rho'_2(\mu'_1 x \otimes y) + \rho'_2(x \otimes \mu'_1 y) + \mu'_1 \rho'_2(x \otimes y) + \mu'_2(\rho'_1 x \otimes \rho'_1 y) + \rho'_1 \mu'(x \otimes y) = H^{-1} W \rho'_2(d_{\mathrm{BN}} x \otimes d_{\mathrm{BN}} y).$$

The modified A_3-relation, applied to the θs, is

$$\begin{aligned}
&\rho'_3(\mu'_1\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}) + \rho'_3(\theta_{ij} \otimes \mu'_1\theta_{jk} \otimes \theta_{ki}) + \rho'_3(\theta_{ij} \otimes \theta_{jk} \otimes \mu'_1\theta_{ki}) + \mu'_1\rho_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}) \\
&\quad = \rho'_2(\mu'_2(\theta_{ij} \otimes \theta_{jk}) \otimes \theta_{ki}) + \rho'_2(\theta_{ij} \otimes \mu'_2(\theta_{jk} \otimes \theta_{ki})) \\
&\quad + \mu''_2(\theta'_{ij} \otimes \rho'_2(\theta_{jk} \otimes \theta_{ki})) + \mu'_2(\rho_2(\theta_{ij} \otimes \theta_{jk}) \otimes \theta'_{ki}) \\
&\quad + \rho'_1\mu'_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}) + \mu'_3(\theta'_{ij} \otimes \theta'_{jk} \otimes \theta'_{ki}) \\
&\quad + H^{-1}W\left(\rho'_3(d_{\mathrm{BN}}\theta_{ij} \otimes d_{\mathrm{BN}}\theta_{jk} \otimes \theta_{ki}) + \rho'_3(d_{\mathrm{BN}}\theta_{ij} \otimes \theta_{jk} \otimes d_{\mathrm{BN}}\theta_{ki}) + \rho'_3(\theta_{ij} \otimes d_{\mathrm{BN}}\theta_{jk} \otimes d_{\mathrm{BN}}\theta_{ki})\right) \\
&\quad + H^{-2}W^2\rho'_3(d_{\mathrm{BN}}\theta_{ij} \otimes d_{\mathrm{BN}}\theta_{jk} \otimes d_{\mathrm{BN}}\theta_{ki}).
\end{aligned}$$

We aim to show that only the fourth line can contribute. The proposition for p' follows.

- The first line is zero because each θ is a ∂-cycle and $\mu_1' = 0$ on $\mathrm{CS}(t_i\bar{t}_i)$.
- The second line is the most difficult. We have

  $$\mu_2'(\theta_{ij} \otimes \theta_{jk}) = bH^{a_{ijk}}\theta_{ik} + dw + \text{lower order cycles.}$$

  We show that each of these terms cannot be part of a contribution. The first one: $\rho_2(\theta \otimes \theta)$ cannot contribute by grading. Next, observe that

  $$\rho_2(dw \otimes \theta_{ki}) = \mu_2'(\rho_1(w) \otimes \theta_{ki}') + WH^{-1}\rho_2(d_{\mathrm{BN}}w \otimes d_{\mathrm{BN}}\theta_{ki}).$$

  If $w \in \ker(\rho_1)$ then we are done. If not, then Lemmas 9.1 and **??** apply. Therefore the dw term cannot be part of a contribution. Finally, the lower order cycles cannot be part of a contribution by the filtration rule.
- The third line is the subject of Lemma 9.1.
- The last two lines cannot contribute by consideration of the homological grading.

This completes the proof for q'. Observe that $q'(\mathbf{t}) + q(\mathbf{t})$ is a coefficient of

$$\mu''(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})$$

and so its invariance can be proved in exactly the same argument as above. Therefore $q(\mathbf{t}) + q'(\mathbf{t})$ is an invariant of the trisection class of $\mathbf{t}$. It follows that $q(\mathbf{t})$ is as well. □

examples: unknotted sphere, projective plane, torus to show asymmetry.

9.3. **Stabilization.** The rest of this section is devoted to proving the following theorem.

Theorem 9.9. *Let $\mathbf{t}'$ be a stabilization of $\mathbf{t}$. Then $q_{ijk}(\mathbf{t}') = q_{ijk}(\mathbf{t})$.*

q' is *not* invariant under stabilization. The best way to see this is to try to prove the theorem for q'. We will see that the difference between $q'(\mathbf{t})$ and $q'(\mathbf{t}')$ comes entirely from Bar-Natan contributions. This will prove Theorem 9.9 and show how to generate examples of q's non-invariance.

Stabilization is not symmetric in the tangles of $\mathbf{t}$. We will study q_{123} and distinguish between *central stabilization* – stabilization which splits a component of $t_2\bar{t}_3$ – and *edge stabilization* – stabilization which splits a component of $t_1\bar{t}_2$ or $t_3\bar{t}_1$. Central and edge stabilization can be defined analogously for q_{231} and q_{312}.

9.3.1. *Central stabilization.* Let $\mathbf{t}'$ be the tri-plane diagram given by splitting a crossingless component in $t_2\bar{t}_3$. Write Θ and Θ' for Θ_{1231} and Θ_{1231}', respectively. Let H be the hyperbox underlying the computation of $\mu_3'(\Theta)$: it is two-dimensional with shape (b, b). $\mu_3'(\theta)$ is the sum of b^2 maps applied to Θ, one for each diagonal of H.

Likewise, H', the hyperbox underlying $\mu_3'(\Theta')$, has shape $(b+1, b+1)$, and $\mu_3'(\Theta')$ is the sum of $(b+1)^2$ maps applied to Θ'.

The arc of stabilization can be specified by the Ξ arcs which lie directly above it. Suppose that these are the n_1-st (on the left) and n_2-st (on right) handles. Call the (n_1+1)-st and the (n_2+1)-st Ξ handles in $\mathbf{t}'$ the *new* handles. See FIGURE. Now we can divide the squares of $H_{\theta'}$ into four groups. We identify each square with its the coordinates of its bottom-left point. Consider the (j_1, j_2) square.

- If $j_1 \leq n_1$ and $j_2 \leq n_2$, then the square is *pre-stabilization.*
- If $j_1 > n_1 + 1$ *and* $j_2 > n_2 + 1$, then the square is *post-stabilization.*
- If $j_1 > n_1 + 1$ *or* $j_2 > n_2 + 1$ but not both, then the square is a *side square.*
  Pre-stabilization, post-stabilization, and side square are all called *old* squares.
- The unique square with $j_1 = n_1 + 1$ and $j_2 = n_2 + 1$ is called the *square of stabilization.*
- The other squares with either $j_1 = n_1 + 1$ or $j_2 = n_2 + 1$, but not both, are called *new* squares.

We will compare $\mu_3'(\theta')$ to $\mu_3'(\theta)$ by showing that new squares and the square of stabilization contribute nothing to $\mu_3'(\theta')$, while the other squares make the same contribution times H. (Accordingly, the squares could be put into two categories: old and new. We find this separation clearer.) Let p'_{j_1,j_2} be the path which uses the diagonal in the square with lower left corner at (j_1, j_2). Conflate this path with the map along this path.

Suppose that p'_{j_1,j_2} is pre-stabilization. Observe that the *active part* of the configuration at $H'_{(j_1,j_2)}$ is identical to that of $H_{(j_1,j_2)}$. Write $\not{p}_{j_1,j_2}$ (resp. $\not{p}'_{j_1,j_2}$) for the composition of all the maps in p_{j_1,j_2} (resp. p'_{j_1,j_2}) up to and including the diagonal. From the observation it follows that

$$\not{p}'_{j_1,j_2}(\theta') = \not{p}_{j_1,j_2}(\theta) \otimes v_+$$

where the extra v_+ factor can be identified with the new component of $t_2'\bar{t}'_3$. (This component is always passive in every configuration in $\not{p}'$.) In short, the topology of the active parts of all the relevant diagrams is the same for the two $\not{p}$ maps. The maps after the diagonal are all single handle attachments. FIGURE shows that the only difference between all the configurations for $p_{j_1,j_2}(\theta)$ and $p'_{j_1,j_2}(\theta')$ after the diagonal is that p'_{j_1,j_2} contains an extra merge and then split on a v_+-labeled, crossingless component. It follows that

$$\langle p'_{j_1,j_2}(\Theta'), \Theta'_{ii}\rangle = U\langle U p_{j_1,j_2}(\theta), \Theta_{ii}\rangle.$$

Suppose that p'_{j_1,j_2} is post-stabilization. The same analysis applies. The only difference is that the extra merge and split occur before the diagonal.

Suppose that p'_{j_1,j_2} is a side square. The analysis is more or less the same. There is an extra merge before the diagonal, and an extra split after. The key

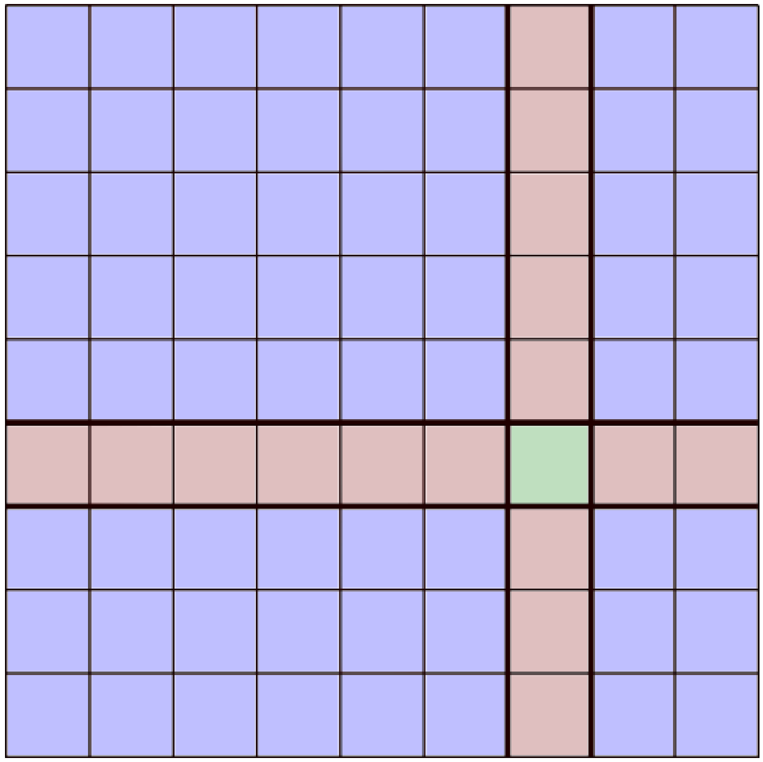

FIGURE 17. The division of H' into old and new squares after a $(?, ?)$-stabilization. The blue squares are old, the red squares are new, and the green square is the square of stabilization.

observation is that the configuration H_{j_1,j_2} does not involve one of the new handles and is still unaffected by the stabilization. We conclude that

$$\langle p'_{j_1,j_2}(\Theta'), \Theta'_{ii}\rangle = U\langle p_{j_1-1,j_2}(\theta), \Theta_{ii}\rangle.$$

or

$$\langle p'_{j_1,j_2}(\Theta'), \Theta'_{ii}\rangle = U\langle p_{j_1,j_2-1}(\theta), \Theta_{ii}\rangle.$$

depending on the side.

Suppose that p'_{j_1,j_2} is new. There are a few possibilities here. If the square is new and left or below the square of stabilization, then it involves a degree one, plus-labeled circle. Therefore $p'_{j_1,j_2} = 0$. If the square is on the upper right, then it involves a dual degree one arrow on a plus-labeled circle. This could support either a Szabó configuration of type E or a dual tree configuration. The type E configuration cannot contribute to the coefficient of the highest generator because the result has a minus-labeled strand at a basepoint. *However, there can be higher Bar-Natan contributions.* See the next section for an example.

Suppose that p'_{j_1,j_2} corresponds to the square of stabilization. The active part of this configuration contains a v_+-labeled circle with in- and out-degree 1. It follows that the diagonal map is zero.

9.3.2. *Edge stabilization.* We can do the same sort of analysis using FIGURE.

Suppose that p'_{j_1,j_2} is pre- or post-stabilization. The analysis is basically the same. The only difference is that the extra-merge-split combo do not happen to the same component. The merge must be of a plus-labeled circle. The split must be as well: there must be a canonical generator with plusses there, and θ_{ii} must have a v_+ there. Therefore

$$\not{p}'_{j_1,j_2}(\theta') = \not{p}_{j_1,j_2}(\theta) \otimes v_+$$

Suppose that p'_{j_1,j_2} is new. Again, there are two possibilities. One involves a degree one arrow on a plus-labeled component and so it cannot contribute. The other involves a dual degree one arrow to a plus-labeled component, and the analysis above applies.

inshallah

Suppose that p'_{j_1,j_2} corresponds to the square of stabilization. The active part of this configuration involves a degree one, plus-labeled circle and therefore cannot contribute.

□

10. Disjoint union and spun $(2,p)$-torus links

If $c_1 = b$, then $q(\mathbf{t}) = 0$ by unitality. (M-Z show that $\mathbf{t}$ is unknotted sphere anyway.)

10.1. **Disjoint union.** Let $\mathbf{t}$ be a split diagram, $\mathbf{t} = \mathbf{t}_0 \cup \mathbf{t}_1$. Then $q(\mathbf{t}) = H^{b_1 - \chi(K_1)/2} q(\mathbf{t}_0) + H^{b_0 - \chi(K_0)/2} q(\mathbf{t}_1)$.

Proposition 10.1. *Let $\mathbf{t} = \mathbf{t}_0 \coprod \mathbf{t}_1$ be a split tri-plane diagram. Then*

$$q(\mathbf{t}) = q(\mathbf{t}_0) + q(\mathbf{t}_1).$$

This is a really bizarre formula. Sort of concordance-y? a knot and its push-off "cancel" because they cobound...

is there a non-split example? spun $(2, 2p)$?

10.2. **Connected sum.** next time

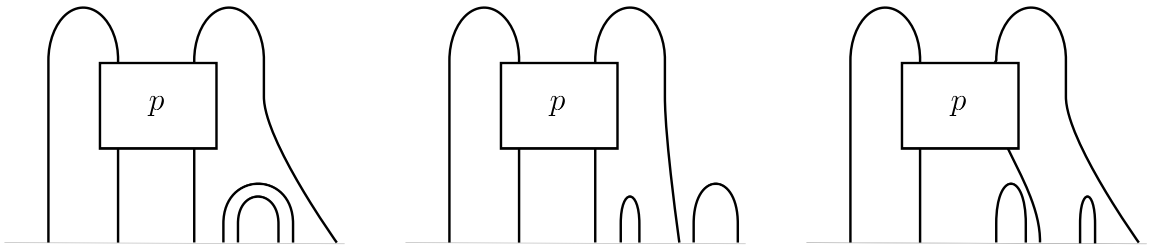

FIGURE 18. The tri-plane diagram $\mathbf{t}_p$ for the spun $(2,p)$-torus link.

10.3. **Spins.** In [6], Meier and Zupan systematically produce bridge trisection diagrams for spins of knots. Figure 18 shows a tri-plane diagram $\mathbf{t}_p$ for the spun $(2,p)$-torus link. If p is odd then $\mathbf{t}_p$ presents a knotted sphere. If p is even and non-zero then $\mathbf{t}_p$ presents a non-trivial link of an unknotted sphere and torus. These diagrams have $b = 4$ and $\chi = 2$.

Theorem 10.2. $q(\mathbf{t}_p) = 1$.

This theorem shows that q can distinguish knotted and unknotted spheres. The method of computation may be widely applicable. We will compute $q(\mathbf{t}_0)$, then show that $q(\mathbf{t}_p) = q(\mathbf{t}_0)$ for all p. In other words, the "braidlike resolution" of $\mathbf{t}_p$ is the only resolution which contributes. It is interesting to consider the topological meaning of such a resolution.

Let's begin with a few observations about computing $q(\mathbf{t})$ in general. A resolution of $t_i\bar{t}_j$ may be written $I\bar{J}$ where I is a resolution of t_i and J is a resolution of t_j. Consider the cancellation

$$t_i\bar{t}_jt_j\bar{t}_k \to t_i\bar{t}_k.$$

Let $x \in \mathrm{CS}(t_i\bar{t}_jt_j\bar{t}_k)$ be a simple tensor in resolution $IJ\bar{J}'K$. Studying the Reidemeister 2 maps (Figure WHICH of [3]), we see that the corresponding map

$$\mathrm{CS}(t_i\bar{t}_jt_j\bar{t}_k) \to \mathrm{CS}(t_i\bar{t}_k)$$

vanishes unless $J = J'$. To compute $\mu_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})$ one must compute representatives for the θs, build a bunch of hyperboxes, and so on. What this discussion shows is that we only need to consider canonical generators in which θ_{ij} and θ_{jk} have mirrored underlying resolutions on t_j. The same argument applies to the θ_{jk} and θ_{ki} with respect to t_k and to θ_{ij} and θ_{ki} with respect to t_i. In other words, we need only consider terms in $\bar{\mu}_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})$ with underlying resolution of the form

$$I\bar{J}J\bar{K}K\bar{I}.$$

Moreover, $\bar{\mu}_3$ is increasing in the underlying resolution of simple tensors. It follows that we need only consider terms of $\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}$ of the form

$$I\bar{J} \otimes J\bar{K} \otimes K\bar{I}.$$

The upshot is that we can basically ignore the "filtered structure" on CS and just look at a few hyperboxes. Lastly, recall that by the filtration rule we may throw out any canonical generators with a v_--label on a bridgepoint.

improve

Proof. We show that $q(\mathcal{K}_0) = 1$ by direct computation in the next lemma.

Observe that each $t_i\bar{t}_{i+1}$ can be unwound to a crossingless diagram using only R2 moves. The canonical generators are therefore of one of the forms shown in Figure **??**. Label the terms by their resolutions, i.e.

$$\theta_{ij} = \sum \theta_{ij;I\bar{I}}.$$

Following the discussion above,

$$\bar{\mu}_3(\theta_{12} \otimes \theta_{23} \otimes \theta_{31}) = \sum_I \mu_3(\theta_{12;I}, \theta_{23;I}, \theta_{31;I}).$$

So we need only consider p products instead of 2^p.

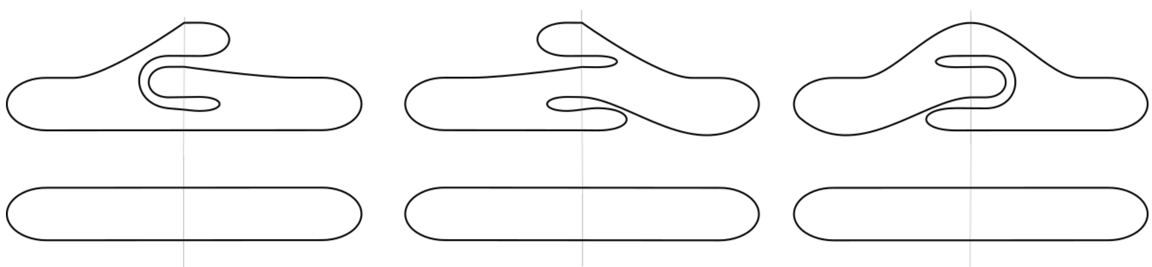

FIGURE 19. $\theta_{12} \otimes \theta_{23} \otimes \theta_{31}$ for $\mathbf{t}_0$. Every component is v_+-labeled. For $p \neq 0$, this generator still appears in the resolution called I_0.

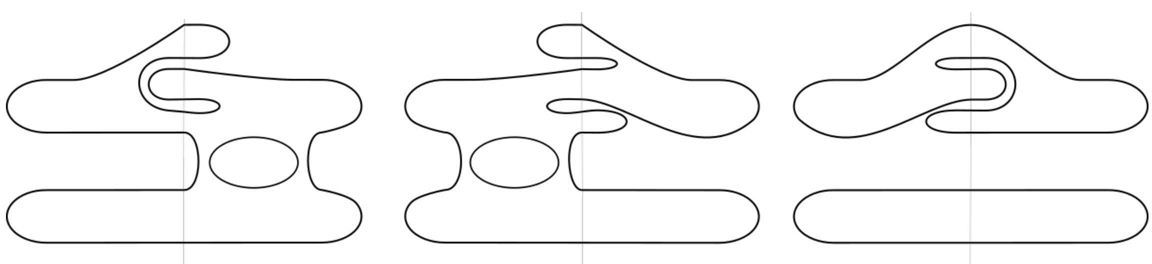

FIGURE 20. $H^{-1}(\theta_{12} \otimes \theta_{23} \otimes \theta_{31})$ for a resolution of $\mathbf{t}_p$ which differs from I_0 only in two crossings. (We have supressed a power of H.) A *type B resolution* is one which differs from I_0 in at most two crossings *in at least one tangle.*

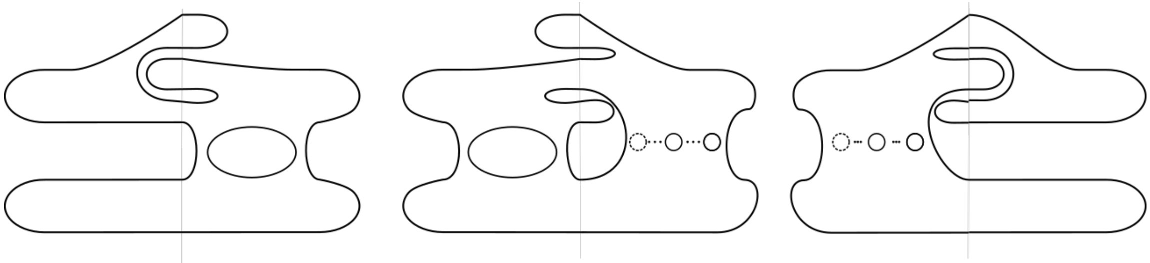

FIGURE 21. An H-multiple of $\theta_{12} \otimes \theta_{23} \otimes \theta_{31}$ for a resolution not considered above. A *type C resolution* is one which differs from a type B resoltuion by changing more crossings. Therefore these generators are formed by applying a Reidemeister 2 move to non-bridge components of a type B or type C resolution. The dotted circle indicates that the circle is v_--labeled. The other possible generators on the same resolution are obtained by swapping the dotting to the opposite circle or removing the dotting and multiplying by H.

$\mathbf{t}_0$ is a resolution J of $\mathbf{t}_p$ for any p. We claim that this is the only resolution which contributes to $q(\mathbf{t})$. The point is that the Reidemeister 2 cancellation requires the central circle to be v_--labeled. Therefore a type B resolution cannot contribute.

For the type C resolutions we must be a little more careful. The situation before attaching canonical handles is shown on the left in FIGURE. On the right is the situation after attaching the handles – the long oval must be v_--labeled because it's the first circle to be capped off in cancellation. We have filled in the circles which are "innermost" among the circles created by the initial Reidemeister 2. These circles must be v_+-labeled. In the penultimate cancellation, these circles are merged. The result is a v_+-labeled circle. This circle is the last to be canceled. We conclude that the final cancellation map comes to zero.

In summary,

$$\langle \mu_3(\Theta_{12}, \Theta_{23}, \Theta_{31}), H^2 W \Theta_{11} \rangle = \langle \mu_3(\Theta_{12;J}, \Theta_{23;J}, \Theta_{31;J}), H^2 W \Theta_{11} \rangle = 1.$$

□

Lemma 10.3. $q(\mathbf{t}_0) = 1$.

Proof. $\mathbf{t}_0$ is the split union of an unknotted sphere and a diagram $\mathbf{t}'$ of a torus. We may ignore the sphere component. The hyperbox for the torus computation is below. In Section 9.3 we saw that the computation comes down to studying the diagonals of each squares of this hyperbox. Any square with a degree one circle contributes nothing: that circle must be v_+-labeled, so the corresponding Szabó map vanishes. That leaves two configurations. One is of type 12 in Szabó's numbering and therefore vanishes. The other (in the top right) is of type 1 and therefore is non-vanishing on $v_+ \otimes v_+$. It follows from a short computation (or our study of the gradings above) that the path with this diagonal contributes a Θ_{11}-coefficient of H^2W.

□

10.4. **Stabilization non-invariance of μ_3'.** stabilize twice(?) at the bottom, then top

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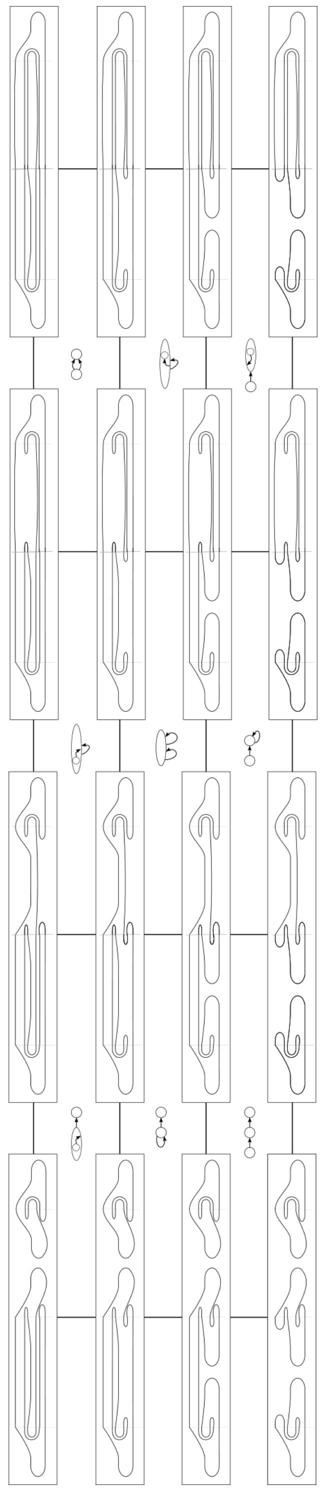

FIGURE 22. The hyperbox from Lemma 10.3. The diagrams between vertices show the corresponding Szabó configuration (up to isotopy).